Commitment and Investment Distortions Under Limited Liability[‡]

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We study how frictions originating from limited liability distort firms' investment and financing choices. By financing new investments with debt, firms can use limited liability to credibly commit to defaulting earlier—allowing both firm owners and new creditors to benefit from diluting existing creditors. In a dynamic setup, this leads to time-inconsistency, increasing the cost of external funds and discouraging investment. We show that the interaction of these two forces leads to heterogeneous investment distortions, where highly-indebted firms overinvest and those with low levels of debt underinvest. Allowing firm owners to pay themselves directly from new debt issuance can mitigate overinvestment but, in the presence of repeated investment opportunities, tends to exacerbate underinvestment among low-leverage firms.

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1 Introduction

Does investment flow to its most efficient uses when firms are burdened with high debt? While existing debt does not distort investment, financing, or payout decisions in the absence of market imperfections, a large and diverse literature points to the empirical and theoretical role of financial frictions. This paper develops a model where limited liability, without any additional frictions, leads to distortions in real investment by changing the timing of default.

While limited liability is an essential, but often implicit, feature of almost all models of financial frictions, it is rarely discussed in isolation.[1] Our paper shows how this common element leads to financial frictions with rich implications for real investment, leverage, and equity payouts. Given that limited liability is a universal institutional feature, by investigating this friction alone, we identify a common set of distortions faced by all firms in the economy—in contrast to distortions implied by more specific types financial contracts, which may be more individually appropriate for subsets of firms (see Lian and Ma (2021)). In our analysis we consider firm owners' financing options and allow for variable investment and equity payouts (such as dividends or equity buybacks). These ingredients prove to be crucial for our results and lead to non-monotone and heterogeneous impacts of outstanding debt on real investment, and—unique among these models of real financial frictions—equity payouts.

To isolate the role of limited liability, we start with a simple baseline model with a single financing-investment opportunity and three types of agents (based on Leland (1994)). In our model, firm owners (i.e., equity holders) operate the firm and are protected by limited liability, preexisting debt investors hold existing debt but are otherwise inactive, and new debt investors competitively price new debt issued by the firm. Firm owners simultaneously choose how much to invest, whether to finance this new investment with debt or their own funds, and how much to directly pay out to themselves. New creditors price debt based on these policies, taking into account that some of the funds may be directly transferred to the firm owners—through equity buybacks or direct dividends—rather than into firm

[1]For example, in models of private information (e.g., Bernanke et al. (1999) or Clementi and Hopenhayn (2006)) the limited liability of firm owners constrains the possible punishments for misreporting. Similarly, in models of inalienable human capital (Kiyotaki and Moore (1997), and Albuquerque and Hopenhayn (2004)) limited liability distorts investment because the lender cannot operate the firm in default, and the equity holder cannot commit to maintain operations. Models of risk-shifting and entrepreneurship (e.g., Jensen and Meckling (1976) or Vereshchagina and Hopenhayn (2009)) highlight the incentives for risk-taking due to the asymmetric payoff structure implied by limited liability. Finally, in models of limited enforcement (Buera et al. (2011), Moll (2014), among many others) firms can abscond with profits or capital at the time of default, and limited liability prevents effective punishments.

assets. Cash flows are assumed to evolve according to a continuous stochastic process and firm owners can walk away from the firm at any time, in which case debt holders claim the remaining assets.

We show that with a single investment opportunity, a firm with preexisting debt has an incentive to invest more than the first-best and finances its investment fully with debt. This conclusion stands in stark contrast to Myers (1977) and the literature on debt overhang (see Hennessy (2004) and Diamond and He (2014) and references therein). The incentive to overinvest arises due to "double-selling" of promised earlier cash flows. By financing new investment with debt, the firm owners increase leverage and commit to defaulting earlier. This transforms some of the coupon payments promised to the existing debt holders into new bankruptcy claims, which can be partially sold to new debt holders. The incentive to dilute preexisting debt holders' coupon claims increases firm owners' marginal benefit of debt financing, leading to overinvestment and, if permitted, direct equity payouts.

Our results contrast with the findings of DeMarzo and He (2021) who, under a slightly different set of assumptions, find that equity holders underinvest. In particular, DeMarzo and He (2021) assume that equity holders make investment decisions continuously and that the recovery value in bankruptcy is zero, whereas we assume that investment-financing opportunities arrive infrequently and the recovery value in bankruptcy is positive. While their assumptions simplify the analysis and lead to an elegant characterization of equity holders' optimal choices, they are at odds with the empirical evidence. Most studies find that default costs are relatively modest (see, for example, Andrade and Kaplan (1998) or Bris et al. (2006)). In addition, it is well-documented that firms' investment decisions are infrequent with a majority of firms making no or little investment from year to year (see, for example, Gourio and Kashyap (2007), Khan and Thomas (2008), or Winberry (2021)).[2]

The severity of equity payout restriction plays an important role in our analysis. In our model with a single investment-financing opportunity, a policy of allowing firm owners to pay themselves from newly issued debt is unambiguously socially desirable because it mitigates inefficient overinvestment.[3] This is because equity payouts provide a more efficient way to increase firms' indebtedness. Thus, all firms, irrespective of their indebtedness, limit

[2]Since dilution is driven by a change in default timing, it is different from standard debt dilution in two-period models (Fama and Miller (1972)). In contrast to risk-shifting and entrepreneurship models (e.g., Jensen and Meckling (1976) and Vereshchagina and Hopenhayn (2009)), debt in our model can be thought of as being collateralized which distinguishes the mechanism from standard channels of dilution. Moreover, the incentive to overinvest in our model also applies to firms that are still well away from their default threshold, not just those currently financially distressed.

[3]The assumption that equity payouts are generally constrained is consistent with Billett et al. (2007) and Chava et al. (2019) who report that covenants restricting both equity payouts and debt issuance are common, particularly among below investment grade and unrated firms.

their inefficient investment by switching to debt-financed equity payouts. In the corner case with unconstrained equity payouts we can split the firm owners' problem into two separate problems: (1) investment and (2) dilution of existing debt holders. Firm owners choose investment to maximize the net present value of the firm and then choose the level and financing of equity payouts to optimally dilute existing debt holders. Allowing firms to hold cash does not mitigate these distortions since firm owners would never choose to have positive cash holdings. The reason is that while firm owners can still dilute existing claims to coupons, there is no way for firm owners to benefit from that dilution when the proceeds are invested in cash (since the cash is dispersed to debt holders in bankruptcy).

We next incorporate repeated financing-investment opportunities into our model, revealing the importance of dynamic considerations. We model repeated financing-investment opportunities with a fixed stochastic arrival rate. When the arrival rate is non-zero, a new channel emerges as debt investors anticipate firm owners' lack of commitment not to dilute their coupons in the future by changing the default timing, raising the cost of debt financing at the time of investment. The resulting incentive to underinvest – defined as foregoing investments whose marginal returns exceed their marginal costs – is particularly relevant for firms with low initial indebtedness and a high arrival rate of future financing opportunities, because these firms have the largest capacity to dilute the coupons of existing debt holders and frequent opportunities to do so. Therefore, we obtain the prediction that firms with low levels of debt tend to underinvest while high-indebted firms—just as in the one-shot model—tend to overinvest. This contrasts predictions from models with collateral constraints, where financially constrained firms protected by limited liability always underinvest (e.g., Buera (2009), Moll (2014), Khan and Thomas (2013), or Buera et al. (2015)).

We investigate the effects of a policy that restricts direct equity payouts from debt issuance in a calibrated version of the repeated model. We find that restricting equity payouts has new heterogeneous implications for the efficiency of investment across firms in our model. By mitigating debt investors' concerns about future dilution, equity payout restrictions lower the cost of debt finance for firms with low levels of debt. This raises these firms' investment, which tends to move their investment closer to the efficient level. In contrast, for highly indebted firms with infrequent financing opportunities, restricting equity payouts exacerbates inefficient overinvestment as in the baseline model with a single financing opportunity.

Taken together, our results indicate that firms protected by limited liability have an incentive to overinvest because debt-financed investment allows them to (1) dilute current debt holders' coupon claims by increasing indebtedness and bringing forward bankruptcy; or (2) limit the gains from investment to debt holders from new investment if increasing indebtedness is too costly. At the same time, our model predicts that firms with low levels of debt and with frequent investment opportunities tend to underinvest. This is because these firms have the largest capacity to dilute the coupons of debt holders in the future, which is anticipated by creditors who require high compensation for lending to those firms. Facing a high cost of debt, these firms cut their investment below its efficient level. These mechanisms are strengthened when firm owners can deplete equity by making discretionary equity payouts to themselves, such as equity repurchases.

Literature Review —

Our paper contributes to the large literature that studies how debt distorts firms' investment decisions when equity holders are protected by limited liability. This literature, initiated by Myers (1977), showed that firms with existing debt tend to underinvest (see Hennessy (2004) or Diamond and He (2014) and references therein). We show that when investment can be financed with debt and investment opportunities arrive infrequently, firms may overinvest. As such, our paper is related to Hackbarth and Mauer (2012) who show that firms may execute an investment opportunity earlier, but do not consider repeated investment opportunities.

Our paper builds on the recent literature that investigates firms' dynamic capital and investment choices. This literature received a recent impetus following Admati et al. (2018) who showed that firms have strong incentives to ratchet up leverage. The indirect dilution in our model operates through changes in the timing of default as opposed to the classical direct dilution of bankruptcy claims as emphasized by Fama and Miller (1972). DeMarzo and He (2021) were the first ones to point out the indirect dilution channel and consider its impact on leverage dynamics and debt issuance policy.

In contrast to DeMarzo and He (2021), we assume that the bankruptcy value of the firm is positive and that firms' investment-financing opportunities arrive infrequently. With these assumptions, we find that equity holders have an incentive to overinvest, while in DeMarzo and He (2021) equity holders always underinvest. To understand why this is the case, note that, on the margin, investing more and financing investment with debt allows firm owners to increase leverage and bring bankruptcy forward. Therefore, as in DeMarzo and He (2021), increasing leverage leads to indirect dilution. However, this is not enough to encourage overinvestment. For overinvestment, it is crucial that increasing leverage transforms some of the coupon payments promised to the existing debt holders into new bankruptcy claims, which can be partially sold to new debt holders. In other words, by increasing leverage equity holders are able to sell again some of the cash flows they promised before to the existing debt holders ("double selling"). It is only through this transformation of claims and their sale to new creditors that indirect dilution directly benefits firm owners and leads to overinvestment. Indeed, it is easy to show analytically in our model that if the recovery value is zero as in DeMarzo and He (2021) then the firm owners underinvest. Thus, our paper is broadly related to recent papers that investigate how modifying various assumptions of DeMarzo and He (2021) affects their implications (see, DeMarzo (2019), Benzoni et al. (2022), and Malenko and Tsoy (2020)). Similarly to Benzoni et al. (2022), we show that infrequent financing opportunities matter, though different from them we study implications for real investment.

We also speak to recent empirical and quantitative literature that investigates investment and corporate leverage at the macroeconomic level (e.g., Atkeson et al. (2017)). Recently, Crouzet and Tourre (2020) investigate how business credit programs can mitigate underinvestment and Acharya and Plantin (2019) argue in a model of agency frictions that equity payouts can lead to underinvestment. Jungherr and Schott (2021) and Kalemli-Özcan et al. (2022) show how high leverage can lead to slow recovery. Finally, Cuciniello et al. (2023) study business start-up subsidies in a general equilibrium framework.

Finally, we contribute to the literature that studies how financial frictions distort firms' investment choices and firms' dynamics. Early contributions to this literature include Cooley and Quadrini (2001), Gomes (2001), and Cooper and Ejarque (2003). Most directly connected to our paper are Albuquerque and Hopenhayn (2004), Clementi and Hopenhayn (2006), and Clementi et al. (2010) who study how financial frictions arising from combinations of limited liability coupled with other imperfections distort firms' investment and affect their growth and how these frictions can be circumvented by optimal dynamic contract (see also Cao et al. (2019)). In contrast to these papers, we focus on limited liability alone and show that even in the absence of additional market imperfections it leads to financial frictions with rich implications for real investment, financing, and equity payouts.

2 Model

There are three types of agents in the baseline model: firm owners (hereafter equity holders) that operate the firm, existing creditors (i.e., debt holders) who hold debt issued in the past, and competitive outside creditors. In our baseline, equity holders face a one-time opportunity at time 0, at which time they can issue new debt and equity, make direct payouts to themselves (i.e., dividends or equity buybacks), and make a real investment. Debt is senior to equity but all debt, including newly issued debt, has equal priority.[4] All actions are perfectly observable and there is complete information. To keep the model

[4]That is, all claims to the assets of the firm in default are pari passu. While this might seem to make the traditional source of dilution possible (i.e., selling new pari passu debt to dilute the claims to the firm in default), by construction that will not be the case in our model. See Section 2.3 for more details.

analytically tractable and to highlight the underlying intuition, in our baseline model we assume that equity holders have one-time opportunity to raise new financing and make a real investment. We extend the model to feature two rounds of investment in Section 3.6 and repeated financing and investment in Section 4.

2.1 Firm State, Notation, and Laws of Motion

The state of a firm is summarized by its *cash flows*, Z, and the book value of its current *liabilities*, L, defined as the present discounted value (PDV) of all promised cash flows to debt holders.[5] Equity and bond holders discount future payoffs at the same constant rate $r > 0$. In the absence of new investment, $Z(t)$ follows a geometric Brownian motion with risk-neutral drift μ and instantaneous volatility of $\sigma^2 > 0$

$$\mathrm{d}Z(t) = \mu Z(t)\mathrm{d}t + \sigma Z(t)\mathrm{d}\mathbb{W}(t), \quad Z(0) > 0, \tag{1}$$

where $\mathbb{W}(t)$ is a standard Brownian motion. Liabilities, L, may have a one-time jump at time 0 (i.e., at the time of investment if equity holders decide to finance some of the investment with debt) but remain constant for all $t > 0$. Thus, we do not allow equity holders to issue new debt or repurchase existing debt after time 0—and relax this assumption in Section 4.

Real Investment and Payouts to Equity At time 0 equity holders have a one-time financing-investment opportunity that expires immediately if not executed. In particular, at time 0 equity holders can deterministically increase initial cash flows of the firm from $Z(0)$ to $\hat{Z} \equiv (1+g)Z(0)$, where $g \geq 0$ captures equity holders' investment financed through a combination of new debt and equity. After the initial jump in cash flows, cash flows follow (1). Investment is costly, with its cost being proportional to Z and given by $q(g)Z \equiv \frac{\zeta g^2}{2} Z$.

At the time of the investment, we also allow equity holders to make direct payouts to themselves, M. We interpret M as equity buybacks, dividends, or leveraged buyouts financed by issuing new debt. Thus, M captures any payout to equity holders that equity holders can use immediately for consumption. The presence of M allows us to consider proposals to limit share buybacks or dividend payments. We assume that equity holders can only consume $M \in [0, \kappa Z]$, where $\kappa \geq 0$ is the parameter capturing institutional constraints on financing equity buybacks with new debt, with $\kappa = 0$ being our baseline.[6]

[5] Due to the absence of fixed costs in this model, cash flows are equivalent to EBITDA profits and proportional to both the assets-in-place and enterprise value. The book value of liabilities L does not take into account the equity holder's option to default. For example, if the firm's liabilities consist of one unit of defaultable consol that promises coupon c every instant of time then $L = \int_0^\infty ce^{-rs}ds$.

[6] We interpret κ as a restriction of equity holders' choices arising from covenants protecting existing debt holders. Covenants restricting equity payouts and subsequent debt issuance are common

The investment in our model should be interpreted as large investment expenditures in the spirit of Gourio and Kashyap (2007). A large literature shows that both investment and large leverage adjustments are infrequent and exhibit inertia (see, for example, Gourio and Kashyap (2007) and Welch (2004)). This can be because investment opportunities arrive infrequently and are subject to large fixed costs (Khan and Thomas (2008) or Winberry (2021)) or because accessing financial markets is costly (Morellec et al. (2012) and Benzoni et al. (2022)).

Financing and Limited Liability For tractability we assume that all debt takes the form of defaultable consols, which pay one coupon until the firm defaults and represent a proportional claim to the firms' assets in bankruptcy.[7] Equity holders can fund the total cost of investment, $q(g)Z$, and the total equity payouts, M, with their own funds (equity financing), by issuing new debt (debt financing), or any linear combination of them. We denote the proportion of debt financing by $\psi \in [0,1]$. If the firm issues only equity (i.e., $\psi = 0$) then the liabilities of the firm, L, have no jump at time 0. If $\psi > 0$, liabilities jump at time 0. We denote by $\hat{L}$ *post-investment liabilities.*

Equity holders are protected by limited liability. This means that after investment and financing choices have been made, equity holders can choose to default and walk away with nothing at any time, whereupon the firm is taken over by debt holders. Equity holders have deep pockets, that is they have sufficient funds to keep the firm as a going concern, if they so choose, even when promised debt payments exceed firm's cash flows.

Note that we assume that all debt is unsecured and treated equally in bankruptcy. This is consistent with empirical studies which find that about three-quarters of all outstanding corporate debt is unsecured (Rauh and Sufi (2010) and Luk and Zheng (2022)). Moreover, according to current bankruptcy laws, unsecured debt is treated equally in bankruptcy (see, for example, Kanda and Levmore (1994)). Nevertheless, it should be mentioned that even in the presence of secured debt our results go through as long as both old and new debt are both collateralized by separate assets (see Section 2.3).[8]

(see Billett et al. (2007) and Chava et al. (2019)). Alternatively, this restriction can be interpreted as arising from financial regulations that protect existing debt holders. The restriction that $M \geq 0$ is without loss of generality since equity holders would never choose $M < 0$ (which, in the model, corresponds to buying back debt).

[7]In Appendix B.4 we show that our results continue to hold when equity holders' finance their investment with debt of finite maturity (modeled as in, for example, Leland (1998) or Chatterjee and Eyigungor (2012)).

[8]See also Section 5, where we discuss the key assumptions of the model.

Equity Value Let $V(Z, L)$ denote the *post-investment* value of equity (i.e., the value of operating the firm after the investment option was executed) when the current cash flows are Z and current liabilities are L. Similarly, let $V^*(Z, L)$ denote the *pre-investment* value of equity (i.e., the value of operating the firm to equity holders at the time they make their investment decision) when time 0 cash flows are Z and time 0 liabilities are L.[9]

It will prove useful to rescale the value of equity with cash flows. Thus, we denote the post- and pre-investment equity value relative to cash flows as $v(\cdot) \equiv V(\cdot)/Z$ and $v^*(\cdot) \equiv V^*(\cdot)/Z$, respectively. Similarly, we define current leverage as $\ell \equiv L/Z$ and the equity payouts per unit of Z as $m \equiv M/Z$.

Value in Default Upon default the firm is taken over by the debt holders who continue to operate it. However, default may have real costs in the sense that immediately following default the firm's cash flows decrease from Z to $(1-\theta)Z$, where $\theta \in [0, 1]$. The parameter θ captures deadweight costs associated with bankruptcy proceedings and debt holders' lower skill in running the firm. We use $\theta = 0$ as our baseline but show robustness to empirically-plausible alternative values for θ.

While estimates find bankruptcy is costly, most studies find these costs to be relatively modest implying that most of the firm's value is preserved in bankruptcy (see, for example, Andrade and Kaplan (1998) or Bris et al. (2006)). The assumption that firm's bankruptcy value is positive is one of the key assumptions that differentiate our model from DeMarzo and He (2021) (see Section 5 for a more detailed comparison with the recent literature on leverage dynamics).

2.2 Equity Holders' Investment and Default Decisions

Equity holders face the following decisions. First, at time 0, they have to choose how much to invest, g, how much to pay out to themselves, M, and how to finance these choices, ψ. Having made these choices, at each instant of time they need to decide whether to keep operating the firm or default instead.

Investment Decision Given an initial state (Z, L), equity holders choose the financing mix ψ and real investment g to maximize the sum of the post-investment equity value and direct payouts to equity, net of new equity injected into the firm. The post-investment equity value is given by $V((1+g)Z, \hat{L})$, where $\hat{L}$ are the post-investment liabilities. Let K denote the quantity of newly issued bonds to finance investment and equity payouts.

[9] We differentiate between pre-investment and post-investment states when discussing the investment decision, in which case we denote the post-investment states by $\hat{L}$ and $\hat{Z} = Z(1+g)$.

Equity holders take the equilibrium price $P(\cdot)$ for newly issued bonds as given, and solve

$$V^*(Z,L) = \max_{\substack{g\geq 0\\ \psi\in[0,1]\\ 0\leq M\leq \kappa Z}} \Big\{ \overbrace{V(\underbrace{(1+g)Z}_{=\hat{Z}}, \hat{L})}^{\text{Post-Investment Equity}} - \overbrace{(1-\psi)q(g)Z}^{\text{Equity Financed}} + \overbrace{M}^{\text{Payouts}} \Big\} \tag{2}$$

$$\text{s.t. } \underbrace{P(\hat{L}, Z, L, g, \psi, M)}_{\text{Equilibrium Price}} \underbrace{K}_{\text{New Bonds}} = \underbrace{\psi q(g)Z}_{\text{Debt Financed}} + \underbrace{M}_{\text{Equity Payouts}} \tag{3}$$

$$\underbrace{\hat{L}}_{\substack{\text{Post-investment}\\ \text{Liabilities}}} = \underbrace{L}_{\substack{\text{Initial}\\ \text{Liabilities}}} + \underbrace{\frac{K}{r}}_{\substack{\text{Book Value of}\\ \text{Issued Debt}}} \tag{4}$$

and subject to the feasibility of the payoffs in default embedded in L and $\hat{L}$. Equation (3) is the budget constraints. It states that funds raised by the equity holders from issuing K units of bonds at price $P(\cdot)$ can be used for equity payouts, M, or to finance a portion of investment costs, $\psi q(g)Z$.[10] Equation (4) describes the dynamics of firm's liabilities. In particular, it states that the post-investment liabilities is equal to the sum of pre-investment liabilities and the book value of newly issued debt, K/r.

To understand how existing liabilities distort equity holders' investment choices we consider the following first-best benchmark.

Definition 1 (First-Best Investment)**.** *We define the first-best undistorted investment, g^u, as investment that maximizes the net present value of the firm. That is,*

$$g^u(Z) \equiv \arg\max_g \Big\{ \overbrace{V((1+g)Z, 0)}^{\textit{Post-Investment Equity}} - \overbrace{q(g)Z}^{\textit{Equity Financed}} \Big\} \tag{5}$$

The first-best investment maximizes the net PDV of firm's cash flows. It also corresponds to the investment that equity holders would choose if the firm had no preexisting debt and investment had to be fully financed with equity. Thus, the first-best investment maximizes the net PDV of firm's cash flows. Since both the payoffs and costs are linear in Z, we can show that g^u is independent of Z throughout our model. The homotheticity that leads to a constant g^u is not essential, but simplifies the analysis.

It is worth noting that when $\theta = 0$ (as in our benchmark case), the above definition is equivalent to defining first-best investment as investment that maximizes the total value of

[10]This formulation implies that creditors hold debt till firm's default. Note that we could have allowed existing debt to be callable by the firm at the market price at the time of investment since the resulting budget constraint would be equivalent to the budget constraint obtained by combining Equations (3) and (4). We thank a referee for pointing this out.

firm (that is, the sum of debt and equity).[11]

Default Decision Equity holders optimally choose to default when the equity value, $V(Z, L)$, reaches 0. Note that after investment only cash flows fluctuate, and the equity holders' *continuation* problem becomes a standard stopping problem (as in Leland (1994) with liabilities L as an additional state), which is given by

$$rV(Z, L) = Z - rL + \mu Z \boldsymbol{\partial}_Z V(Z, L) + \frac{\sigma^2}{2} Z^2 \boldsymbol{\partial}_{ZZ} V(Z, L) \tag{6}$$

$$V(\underline{Z}, L) = 0 \tag{7}$$

$$\boldsymbol{\partial}_Z V(\underline{Z}, L) = 0, \tag{8}$$

where $\underline{Z}$ is the endogenous default threshold. Here (7) and (8) are the standard value-matching and smooth pasting conditions, respectively. Define

$$\eta \equiv \frac{(\mu - \sigma^2/2) + \sqrt{(\mu - \sigma^2/2)^2 + 2\sigma^2 r}}{\sigma^2} > 0 \tag{9}$$

$$\chi \equiv \left(\frac{(r - \mu)\eta}{\eta + 1} \right)^{\eta} > 0 \tag{10}$$

$$s(\ell) \equiv \frac{\chi}{\eta + 1} \ell^{\eta}, \tag{11}$$

where $s(\ell)$ captures the value of equity holders' option to default per unit of liabilities. We now characterize the solution to equity holders' default problem (6)-(8).

Proposition 1 (Post-Investment Equity and Default)**.** *Suppose that the current state of the firms is (Z, L). Then the equity value of the firm is given $V(Z, L) = v(\ell)Z$, where $\ell \equiv L/Z$ is firm's leverage and*

$$v(\ell) = \frac{1}{r - \mu} - \ell(1 - s(\ell)) \tag{12}$$

The endogenous default threshold $\underline{Z}$ satisfies

$$\frac{\underline{Z}(L)}{r - \mu} = \frac{\eta}{1 + \eta} L \tag{13}$$

[11]When $\theta \in (0, 1]$, the two definitions do not coincide because the latter definition takes into account the deadweight cost of default. Nevertheless, our results continue to hold under either definition and, thus, do not depend on the choice of the first-best investment benchmark.

and the liquidation value per unit of liabilities is given by

$$\frac{V((1-\theta)\underline{Z}(L),0)}{L} = \frac{(1-\theta)\eta}{1+\eta} \tag{14}$$

Proof. See Appendix A.1. □

The above proposition characterizes the value of equity and equity holders' optimal default decision. Note that if equity holders were not allowed to default then the value of equity would be given by $\left(\frac{1}{r-\mu} - \ell\right) Z$ as equity holders would have to repay all their liabilities. Thus, $s(\ell)\ell$ captures the value of equity holders' option to default (scaled by cash flows). In what follows, we assume that $Z_0 > \underline{Z}$.

Proposition 1 also implies that the first-best investment (as defined in Equation (5)) is given by

$$g^u \equiv \arg\max_g \{ \overbrace{(1+g)v(0)}^{\text{Post-Investment Equity}} - \overbrace{q(g)}^{\text{Equity Financed}} \} \tag{15}$$

2.3 Pricing of Debt

Debt is priced by outside creditors who are risk-neutral and who anticipate equity holders' optimal default decision (as characterized in Proposition 1). For tractability, we assume that all debt takes the form of defaultable consols following Leland (1994, 1998). A defaultable consol pays 1 in perpetuity prior to default and receives a share of the bankruptcy value of the firm in default.

Let T denote the stopping time when cash flows first reach default threshold, $\underline{Z}$, at which point equity holders choose to default. Conditional on the current state of the firm (Z, L) and equity holders' optimal default decision, the market price of a such bond $P(Z, L)$ equals

$$P(Z,L) \equiv \frac{p(Z,L)}{r} = \underbrace{\mathbb{E}_T\left[\int_0^T e^{-r\tau}\mathrm{d}\tau\right]}_{\substack{\text{PDV of promised coupons} \\ = P^C(Z,L)}} + \underbrace{\mathbb{E}_T\left[e^{-rT}\frac{V((1-\theta)\underline{Z}(L),0)}{rL}\right]}_{\substack{\text{PDV of claims in bankruptcy} \\ = P^B(Z,L)}}, \tag{16}$$

where $p(Z, L)$ denotes the price relative to the risk-free rate. Equation (16) emphasizes that a defaultable consol consists of pre-bankruptcy component (i.e., the coupon payments prior to default) with market price $P^C(Z, L)$, and bankruptcy claims with market price $P^B(Z, L)$. We use $p^C(\cdot,\cdot)$ and $p^B(\cdot,\cdot)$ to denote these prices relative to the risk-free rate, r.

Proposition 2 establishes that leverage ($\ell \equiv L/Z$) is the relevant state for pricing debt, solves for the prices of the defaultable consol and of its pre-bankruptcy and bankruptcy components, and characterizes equity holders' budget constraint, (3).

Proposition 2 (Pricing of Debt)**.** *The relevant state for pricing debt is leverage* $\ell \equiv L/Z$*. Moreover,*

$$p(\ell) = 1 - (1 + \theta\eta)s(\ell) \tag{17}$$
$$p^C(\ell) = 1 - (1 + \eta)s(\ell) \tag{18}$$
$$p^B(\ell) = (1 - \theta)\eta s(\ell) \tag{19}$$

Given these prices, the budget constraint (3)*, normalized by* Z*, can be expressed as*

$$p(\hat{\ell})\left((1 + g)\hat{\ell} - \ell\right) = \psi q(g) + m \tag{20}$$
$$p(\hat{\ell}) \geq p^B(\hat{\ell}) \tag{21}$$

where $\hat{\ell} \equiv \hat{L}/\hat{Z}$ *is post-investment leverage.*

Proof. See Appendix A.2. □

The normalized budget constraint (20) is standard.[12] The left hand side equals the total value of new debt issued, normalized by Z. The right hand side represents equity holders' need to raise new debt financing, and equals the debt financed portion of investment costs plus equity payouts, also normalized by Z. The constraint (21) enforces the feasibility of the default payoff embedded in defaultable consols, meaning that equity holders cannot issue so much debt that the firm is strictly above its default threshold immediately after investment. While the constraint (21) never binds when $\kappa = 0$, it may bind if direct payments to equity holders are allowed ($\kappa > 0$).[13]

In the case without bankruptcy costs (i.e., $\theta = 0$) the price of the defaultable consol relative to the risk-free rate simplifies to $p(\ell) = 1 - s(\ell)$. In this particularly simple case, $s(\ell)$ can be interpreted as the spread relative to the risk-free rate. Recall that $s(\ell)$ also appears in the expression for $v(\ell)$ (the normalized value of equity) with the opposite sign, where it captures the value of equity holders' option to default. Thus, we see that a more valuable default option increases the value of equity at the expense of bond holders.

[12]The budget constraint results from combining budget constraints (Equation (3)) with equation describing leverage dynamics (Equation (4)).

[13]This constraint is similar in spirit to a constraint that limits the ability of the firm to sell assets in distress but before bankruptcy (as is the case in common bankruptcy laws). It captures the observation that existing debt holders would be able to (at least partially) block issuance of large amount of debt just before default.

Bankruptcy claims Proposition 2 prices an asset which bundles coupon payments with a claim to a fraction of the liquidation value of the firm upon default (i.e., assuming the proportional distribution of claims is usually referred to as pari passu). A concern might be that since there is no seniority of existing claims on the firm in bankruptcy relative to new claims, the firm is able to capitalize on that by issuing so much debt (possibly an infinite quantity) to dilute existing debt holders' bankruptcy claims (see, for example, the discussion in Section 2.D of DeMarzo and He (2021)).

This, however, is not possible in our model. First of all, constraint (21) restricts equity holders to issue debt up to a finite limit implied by the default threshold. Second, for all ℓ that satisfy constraint (21), the value of bankruptcy claims, $p^B(\ell)$ is actually increasing in ℓ. The intuition is that while new debt does increase the number of claimants in bankruptcy, it also induces firms to default at a higher Z. This increases the liquidation value of the firm (as the liquidation value is proportional to L in (14)) and offsets the effect of an increase in the number of claimants in bankruptcy. In addition, since the firm now defaults on average earlier, the present value of existing debt holders' bankruptcy claims actually increases.

Collateralized Debt as an Equivalent Formulation We can equivalently assume that the existing debt is collateralized with the firm's pre-investment assets, without any changes to the equity holders' optimal choices. Since the value of the firm in bankruptcy is known with certainty, we can interpret that value as collateral that equity holders' can use to collateralize debt. We then impose the constraint on equity holders' so that any collateral pledged to existing debt holders cannot be pledged to new debt holders (or more generally, that equity holders' cannot pledge the same collateral to multiple debt holders). Let us denote by $p^S(\cdot)$ the value of existing collateralized debt relative to risk-free interest rate, r. Similarly, let $\hat{p}^S(\cdot)$ denote the new collateralized debt sold to finance investment.

As we show in Appendix A.2, in equilibrium, we have $p^S(\hat{\ell}) = \hat{p}^S(\hat{\ell}) = p(\hat{\ell})$. That is, the equilibrium price of the existing and new collateralized debt is identical, and both are the same as the price of the baseline asset. Moreover, we show that the (21) is implied by the pledgeability constraint that ensures that equity holders' cannot pledge the same collateral to multiple debt holders. Thus, we conclude that equity holders will make the same real investment decisions when debt is collateralized.

3 Analysis of Investment Decision

In this section, we first analyze equity holders' investment and financing decisions in our baseline model with a single financing-investment opportunity. We then discuss briefly several extension including a model with two investment opportunities.

3.1 Sources of Investment Distortions

Let $H(\ell) \equiv \theta\eta s(\ell)$ denote deadweight cost of default per unit of leverage. The investment problem can be characterized as follows.

Proposition 3. *Equity holders' investment problem can be written as*

$$v^*(\ell) = \max_{\substack{g,\hat{\ell}\geq 0 \\ \psi\in[0,1] \\ 0\leq m\leq\kappa}} \left\{ \frac{1+g}{r-\mu} - q(g) - p(\hat{\ell})\ell - (1+g)H(\hat{\ell})\hat{\ell} \right\} \tag{22}$$

$$s.t.\, p(\hat{\ell})((1+g)\hat{\ell} - \ell) = \psi q(g) + m \tag{23}$$

$$p(\hat{\ell}) \geq p^B(\hat{\ell}) \tag{24}$$

The first-best investment, g^u, is the unique solution to

$$0 = \frac{1}{r-\mu} - q'(g^u) \tag{25}$$

If $q(g) = \zeta g^2/2$ then the first-best investment is given by $g^u = \frac{1}{\zeta(r-\mu)}$.

Proof. See Appendix A.3 for details. □

The reformulated objective function (22) shows that, since new debt is fairly priced, equity holders bear the full cost of the investment and any change in the expected deadweight cost of default. For the same reason equity payouts m do not appear directly in (22). However, ψ and m affect the post-investment value of equity indirectly through $\hat{\ell}$. Finally, the constraint (24) enforces the feasibility of the default payoff embedded in defaultable consols, meaning that equity holders cannot issue so much debt that the firm is strictly above its default threshold immediately after investment.

Compared to Equation (15), we see that equity holders face two distortions. The first distortion is due to existing debt, as captured by $p(\hat{\ell})\ell$. Since $p(\hat{\ell})$ is a decreasing function of post-investment leverage, $\hat{\ell}$, equity holders have an incentive to increase leverage. This is the classic conflict between equity and debt holders pointed out by Myers (1977). The second distortion is due to bankruptcy costs. Since $H(\cdot)$ is an increasing function, the presence of bankruptcy costs discourages equity holders from taking on additional leverage.

Equation (22) implies that the FOC that determines investment in the absence of default costs (our benchmark case) is given by

$$\frac{1}{r-\mu} - q'(g) - p'(\hat{\ell})\frac{\partial\hat{\ell}}{\partial g}\ell = 0 \tag{26}$$

Equation (26) is a key equation of our model.[14] Compared to (25), which determines the first-best investment, the FOC includes the additional term $-p'(\hat{\ell})\frac{\partial \hat{\ell}}{\partial g}\ell$. This term captures the marginal change in the value of existing debt due to the change in the firm's distance to default. Since the value of existing debt is decreasing in leverage (i.e., $p'(\hat{\ell}) < 0$) it follows that the sign of this distortion depends on the sign of $\frac{\partial \hat{\ell}}{\partial g}$. If at optimal investment we have $\frac{\partial \hat{\ell}}{\partial g} > 0$ then equity holders *overinvest* relative to first-best. If at the optimal investment we have $\frac{\partial \hat{\ell}}{\partial g} < 0$ then equity holders *underinvest*.

3.2 Dilution Mechanism and Inefficient Investment

We now present our first main result that preexisting debt encourages overinvestment. We first characterize equity holders' choices without equity payouts (i.e., $\kappa = 0$).[15]

Proposition 4. *Given $\kappa = 0$, denote g^* as equity holders' optimal investment*

1. *If constrained to use equity financing, equity holders underinvest ($g^* < g^u$)*

2. *If allowed to choose financing optimally, equity holders: (a) finance all their investment with debt; (b) overinvest ($g^* > g^u$)*

Proof. See Appendix B.2 □

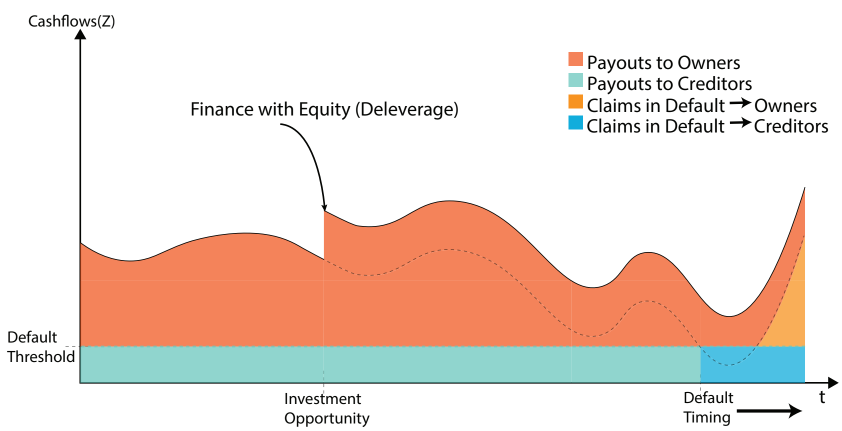

Figure 1: **Equity-financed investment, due to deleveraging, decreases the option value of default.** This figure shows debt and equity cash flows with an equity-financed investment opportunity. We show the simplified case with no bankruptcy costs ($\theta = 0$).

[14] Note that once $\{g, \psi, m\}$ are chosen, $\hat{\ell}$ is determined by the budget condition (23). Thus, we can treat $\hat{\ell}$ as a function of $\{g, \psi, m\}$.

[15] In Appendix B.4 we show that this result (and other results reported in this section) continues to hold when equity holders finance their investment with debt of finite maturity as in Leland (1998).

The first part of Proposition 4 nests the classic underinvestment result of the debt overhang literature (Myers (1977)). Thus, our model makes precise that equity financing is a condition required for this classic result. Figure 1 visualizes the intuition. Investment financed with equity leads to deleveraging, leading equity holders to pay coupons to creditors for longer. As a result, a portion of the cash flows from the new investment is allocated to existing debt holders in form of coupon payments (the portion of the "Claims in Default → Creditors" area above the dotted line). The benefit from new investment is hence partly captured by existing debt holders, implying that equity holders' benefit from investment is lower than the social benefit. In terms of the key equation (26), deleveraging implies that $-p'(\hat{\ell})\frac{\partial\hat{\ell}}{\partial g}\ell < 0$, and hence a reduction of equity holder's incentive to invest.

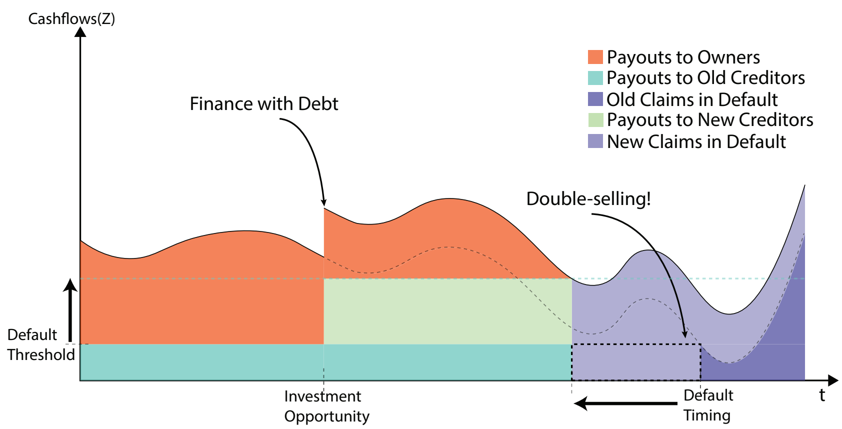

Figure 2: **Debt-financed investment, due to increased leverage, dilutes existing debt holders by double-selling some of their promised coupon payments.** This figure shows debt and equity cash flows with a debt financed investment opportunity. We show the simplified case with no bankruptcy costs ($\theta = 0$).

The second part of Proposition 4 shows that equity holders overinvest if they can choose their financing optimally, provided that the firm has preexisting debt (i.e., $\ell > 0$). Why do equity holders overinvest? This is driven by two related forces: (1) an incentive to dilute existing debtholders (that motivates low- and medium-leverage firms' choices), and (2) the incentive to limit the gain from new investment to existing creditors (that motivates high-leverage firms' choices, for whom costs of dilution exceed its benefits). As we will explain in detail below, in both cases equity holders issues debt (above the socially optimal level) to shorten the expected length of time they have to pay coupons to existing debt holders.

How can equity holders dilute existing debt holders in the model? Figure 2 illustrates that a sufficiently large debt-financed investment leads to a higher leverage and an earlier default, transforming a portion of the coupon payments that has been promised to existing debt holders (the rectangular area with dashed edges) to claims in default, which have to be shared with new debt holders. Thus, by issuing new debt and increasing leverage, equity holders can sell again claims to some of the cash flows that were previously promised to existing debt holders. It follows that the marginal benefit to equity holders of investing exceeds the social benefit, and hence equity holders overinvest relative to the first-best. In terms of (26), this additional benefit is captured by $-p'(\hat{\ell})\frac{\partial\hat{\ell}}{\partial g}\ell > 0$.

It is worth pointing out that only the coupon payments promised to existing debt holders are double-sold by equity holders, not the existing debt holders' bankruptcy claims. In particular, the value of existing debt holders' bankruptcy claims at the time of default is unaffected by the increase in leverage.[16] Furthermore, since after investment default happens on average earlier, the PDV of existing bankruptcy claims actually goes up. Thus, the dilution mechanism in our model operates through changes in the default timing and affects negatively only the value of existing debt holders' coupon claims. As such this mechanism differs from the standard dilution mechanism emphasized in the literature, where equity holders issue an excessive amount of debt (particularly just before default) to dilute existing creditors' claims in default (Fama and Miller (1972)).

A similar mechanism drives overinvestment by firms with high leverage. If initial leverage is sufficiently high equity holders choose to deleverage (i.e., choose $\hat{\ell} < \ell$). This is because at high levels of initial leverage, increasing leverage further requires a very large inefficient investment. The cost of such inefficient investment are borne by the equity holders and exceed the benefits of dilution. However, even if equity holders deleverage they still overinvest. This is because, on the margin, increasing investment above the efficient level and financing it with debt allows the equity holders to limit the amount of cash flows generated by new investment that would be captured by existing debt holders.[17]

The overinvestment identified in Proposition 4 stands in a stark contrast to models of financial frictions featuring collateral constraints. In the latter, limited liability coupled with additional sources of market incompleteness (such as private information in Clementi et al. (2010) or the ability to abscond funds in Buera et al. (2011)) leads to underinvestment by constrained firms (i.e., firms with low assets relative to debt or borrowing needs). In contrast, we show that limited liability by itself provides an incentive to overinvest.

[16] To be precise, *before* the investment takes place, the value of existing debt holders' bankruptcy claims at the time of default is $\frac{\eta}{1+\eta}L$. *After* the investment takes place the value of existing debt holders' bankruptcy claims at the time of default is given by $V((1-\theta)\underline{Z}(\hat{L}),0)\times(L/\hat{L})$ since the post-investment firm's value in default $(\frac{\eta}{1+\eta}\hat{L})$ is divided proportionally between new and old creditors.

[17] Note that deleveraging occurs not because firms buy back debt, but rather because post investment the cash flows increase more than debt (as issuing more debt becomes too costly for high-leverage firms). Thus, just as in the recent literature on leverage ratcheting (Admati et al. (2018)), in our model equity holders never have an incentive to buy back debt.

3.3 Equity Payouts

We now turn to analyzing how debt financed payouts to equity holders, such as dividends and equity buybacks, affect real investment (i.e., $\kappa > 0$). We define $\bar{m}(\psi, g)$ as the highest equity payout that satisfies the budget constraint (23) given investment, g, and financing choice, ψ. Thus, given $\{g, \psi\}$, equity holders' choice of m has to satisfy $m < \min\{\kappa, \bar{m}(\psi, g)\}$. As shown in Appendix B.3 (Equation (B.7)), $\bar{m}(\psi, g) = \frac{1+g}{r-\mu} - \frac{\eta}{1+\eta}\ell - \psi q(g)$.

Proposition 5. *Denote g^*, m^*, ψ^* as the equity holders' optimal choices of investment, payouts, and financing, respectively. There exists $\underline{\kappa} \in \mathbb{R}_+$ such that*

1. *If $\kappa < \underline{\kappa}$ then equity holders: (a) overinvest ($g^* > g^u$); (b) finance investment and equity payouts with debt ($\psi^* = 1$); (c) make payouts to themselves up to the constraint ($m^* = \kappa$); (d) continue operating firm*

2. *If $\kappa \geq \underline{\kappa}$ then equity holders: (a) invest the first-best amount($g^* = g^u$); (b) finance investment and equity payouts at least partially with debt ($\psi^* \in [\max\{\underline{\psi}_\kappa, 0\}, 1]$, where $\underline{\psi}_\kappa > 0$ is the unique solution to $\kappa = \bar{m}(g^*, \underline{\psi}_\kappa)$); (c) make maximal feasible payouts to themselves ($m^* = \bar{m}(g^*, \psi^*)$); (d) are indifferent between defaulting and continuing to operate the firm*

The threshold $\underline{\kappa}$ is decreasing in ℓ and r, and increasing in σ.

Proof. See Appendix B.3. □

Proposition 5 extends our overinvestment result to the case in which equity payouts financed with debt are permitted. As long as equity holders face sufficiently tight restrictions on equity payouts financed by debt ($\kappa < \underline{\kappa}$), we find that they continue to overinvest. Different from the case with $\kappa = 0$, equity holders accompany investment with direct equity payouts, further increasing post-investment leverage.

By contrast, when the constraint on equity payouts is lax ($\kappa \geq \underline{\kappa}$), equity holders invest the first-best amount. In this case, equity holders have a more efficient way of increasing leverage than inefficient investment. Thus, when $\kappa \geq \underline{\kappa}$, equity holders' problem is decoupled into two separate problems: (1) an investment problem and (2) a dilution of existing debt holders problem. Equity holders choose g to maximize the net present value of the firm and m to maximize the transfer from existing debt holders to themselves. The latter implies choosing the highest feasible m so that the firm defaults right after investment. Thus, when

$\kappa \geq \underline{\kappa}$, equity holders essentially sell the firm to the new debt holders.[18]

Proposition 5 shows that restrictions on equity payouts can increase investment and reduce the probability of bankruptcy, in line with the intuition in Myers (1977). Different from Myers (1977), however, we find that such restrictions might not be desirable. We reach different conclusion since in our model investment is debt financed and tends to be inefficiently high. However, we will see in Section 4 that dynamic considerations lead to more nuanced results.[19]

3.4 Cash and Investment Timing

Above we showed that by selling new debt and increasing leverage, equity holders can credibly commit to default earlier. This emphasis on timing leads to some important questions on robustness: Would the investment distortions remain if the firm could hold cash? And if it could hold cash, is it crucial that the timing of investment coincides with the timing of raising funds from the capital markets?

As we show in Appendix B.5, our mechanism is robust to introducing cash into the model since equity holders have no incentive to hold cash inside the firm. Intuitively, a firm which issues debt and saves the proceeds in form of cash continues to have the ability to dilute existing claims to coupons as in Section 3.2. However, the difference is that in this case there is no way to for equity holders to directly benefit from this dilution as cash is added to the assets of the firm and is distributed to debt holders in bankruptcy. The indirect benefit itself is not enough to compensate the equity holders for the cost of raising cash and, thus, equity holders have no incentive to raise cash.

The above discussion also suggests that the assumption that financial and investment decisions occur simultaneously is not important for our conclusions. The key distortions in our model arise as long as equity holders have some way to directly benefit from diluting existing coupon claims. A simple extension of our model could decouple financing and investment opportunities by modeling them as independent Poisson processes. We conjecture that in this case at each investment opportunity the firm would issue debt to raise cash that would then be invested at the next investment opportunity. We think of the reduced-form cost of real investment in our simpler model as capturing a variety of costs, potentially including the cost of holding cash while waiting for the next investment opportunity, so the main forces leading to over- and under-investment should be similar in such a model

[18] This suggests that in our setup, equity holders have an incentive to "collude" with new creditors in order to dilute the existing debt holders. A similar mechanism has been emphasized recently by Aguiar et al. (2019) within the context of sovereign default.

[19] In Appendix E, we also argue that commonly used covenants are unlikely to correct equity holders' incentives.

extension. This logic suggests that the important assumption on the timing in our model is therefore that opportunities to sell new debt are lumpy.

3.5 Bankruptcy Costs

The above analysis abstracts from default costs ($\theta = 0$). In the presence of default costs (i.e., $\theta \in (0,1]$), debt financing is associated with the following trade-off. On the one hand, as before, issuing new debt allows equity holders to resell some of existing debt holders' claims, which we have seen encourages overinvestment. On the other hand, issuing new debt increases the deadweight cost of default and hence the cost of debt financing, encouraging underinvestment.

It is straightforward to show analytically that for sufficiently low θ Proposition 4 and Proposition 5 continue to hold. This is intuitive: when θ is small then the benefit of issuing additional debt exceeds the associated increase in the deadweight cost of default. In contrast, for sufficiently high θ an increase in deadweight cost of default associated with issuance of new debt dominates. As a consequence, in this case equity holders finance their investment with equity and underinvest. In the Appendix B.6, we provide a numerical examples that show how increasing θ affects the extent of overinvesting.

3.6 Two Investment Opportunities

Before moving to the model with repeated investment opportunities, we consider a simple model with two investment opportunities. We use it to show how the forward-looking behavior of new creditors shifts the cost of future overinvestment and dilution onto equity holders and pushes them towards underinvesting. Whether equity holders overinvest or underinvest depends on whether the benefit from diluting existing debt holders exceeds the costs due to a reduction in the price of new debt. The analysis below also helps to clarify the time inconsistency of equity holders' decisions that appear in the model with many investment opportunities.

Setup The equity holders face the first investment opportunity at time 0 and the second investment opportunity at a known time $T > 0$. Thus, the model can be divided into two "periods": the first "period" corresponding to $t \in [0, T)$ and the second "period" corresponding to $t \geq T$ (where the second period is identical to our benchmark model with a single investment opportunity). As in the benchmark model, we assume that at each investment opportunity, equity holders can increase these cash flows to $Z(1+g)$, where $g \geq 0$ denotes equity holders' investment. This investment is costly with the cost of investment g given by $q(g)Z$. The equity holders begin with pre-existing liabilities such that the initial pre-investment leverage is $\ell_0 \geq 0$.

We make two simplifying assumptions. Namely, we assume that for $t \in [0, T)$ cash flows are not-stochastic and follow $dZ_t = \mu Z_t dt$ with $Z_0 > 0$ with a possible jump at time 0 due to investment (while for $t \geq T$ cash flows follow a process specified in (1)) and that equity holders never default before time T. These assumptions substantially simplify analysis but do not change the nature of the main economic forces that drive equity holders' choices. Finally, we focus on the case $\kappa = 0$ and assume that investment is financed with debt. All derivations are relegated to the appendix.

Investment decision at $t = 0$ We focus our analysis on the investment at time 0 as equity holders' choices at time T are the same as in our benchmark model analyzed above.

Let g_T^* be the equity holders' optimal investment at time T (as characterized above); ℓ_0 and $\hat{\ell}_0$ denote the pre-investment and post-investment leverage, respectively, at time 0; and $p_0(\hat{\ell}_0)$ the price of debt at time 0 as a function of post-investment leverage.

Proposition 6. *The equity holders' problem at time* 0 *is given by*

$$\max_{g_0} \left\{ \frac{1+g_0}{r-\mu} \left(1 - e^{-(r-\mu)T}\right) - q(g_0) - p(\hat{\ell}_0)\ell_0 + e^{-(r-\mu)T}(1+g_0) \left(\frac{1+g_T^*}{r-\mu} - q(g_T^*) \right) \right\} \tag{27}$$

$$s.t. \;\; p_0(\hat{\ell}_0)(\hat{\ell}_0(1+g_0) - \ell_0) = q(g_0) \tag{28}$$

Several important observations follow from Equation (27). First, we note that the expression in (27) does not depend on the value of debt issued either at time 0 or time T (only the value of pre-existing debt affects the value of equity). This reflects the forward-looking behavior of creditors. In particular, new creditors understand that in the future, equity holders will dilute their claims and, thus, purchase firm's debt at prices that reflect equity holders' future behavior. Thus, the costs of future overinvestment are borne by equity holders.

Equation (27) also indicates that the costs of future overinvestment are captured by the term

$$e^{-(r-\mu)T}(1+g_0) \left(\frac{1+g_T^*}{r-\mu} - q(g_T^*) \right)$$

This term captures the net PDV of cash flows generated by the firm from time T onwards and, for a given value of g_0, is maximized at the time-T efficient investment, g_T^u. Since equity holders always choose $g_T^* > g_T^u$ (see Proposition 4), the future cash flows are lower than under efficient investment, which decreases the value of equity and also the return to investment at time 0. Furthermore, since the extent of overinvestment at time T is increasing in leverage at time T, this discourages equity holders from leveraging up at time

0. These considerations provide equity holders with an incentive to *underinvest* at time 0.

At the same time, Equation (27) also shows that the value of equity at time 0 decreases in the value of preexisting liabilities, $p_0(\hat{\ell}_0)\ell_0$. Thus, as in the model with a single investment opportunity, equity holders have an incentive to leverage up and *overinvest* at time $t = 0$. It follows that whether equity holders overinvest or underinvest depends on which force dominates. If the benefit from diluting existing debt holders exceeds the costs of future overinvestment then equity holders overinvest, otherwise they underinvest.

Finally, Equation (27) clarifies the time-inconsistency problem faced by equity holders. From time 0 perspective, equity holders dislike inefficient overinvestment at time T as ultimately they bear its full cost. However, at time T, equity holders always find overinvesting optimal. Thus, it would be in equity holders' interest if they were able to commit at time 0 not to overinvest at time T. Note that the root cause of this time inconsistency is limited liability since it is limited liability that provides equity holders with an incentive to increase leverage and overinvest. As in the model with a single investment opportunity, limited liability is also essential as a commitment device that allows equity holders to credibly commit to default earlier (and when the assets distributed in bankruptcy are higher, which allows them to "double-sell" the cashflows).

4 Repeated Financing and Investment

So far, we have analyzed a model with a single financing-investment opportunity. In that setting, we have shown that equity holders have an incentive to finance their investment with debt and overinvest. We have argued that this behavior is driven by the incentive to dilute preexisting debt holders coupons (among low-leverage firms) and to limit the gain from new investment to existing debt holders (among high-leverage firms). In this section, we build on this model and allow for repeated financing-investment opportunities. We show that repeated financing has non-trivial consequences as buyers of new debt price in the likelihood of future dilution, thereby increasing the cost of debt financing. We find that dynamic considerations can lead to underinvestment especially among low leverage firms with frequent financing-investment opportunities, and that direct equity payouts (i.e., equity buybacks and dividends) financed out of debt further exacerbate underinvestment among these firms because they make debt financing more expensive.

4.1 Model with Repeated Investment

We consider the same setup as described in Section 2, but assume that financing-investment opportunities arrive at a constant Poisson rate, λ. As above, the state of the firm at any given point in time is $\{Z, L\}$. Upon arrival of a financing-investment opportunity, equity holders have the choice to increase current cash flows from Z to $Z(1+g)$ at cost $Zq(g)$. It follows that cash flows follow a jump diffusion

$$\mathrm{d}Z(t) = \mu Z(t)\mathrm{d}t + \sigma Z(t)\mathrm{d}\mathbb{W}(t) + g(Z(t^-), L(t^-))Z(t^-)d\mathbb{N}(t), \quad Z(0) > 0, \tag{29}$$

where $\mathbb{N}(t)$ is a Poisson process with intensity $\lambda \geq 0$ and $g(Z(t^-), L(t^-))$ is equity holders' investment at time t conditional on the state of the firm $\{Z, L\}$ and the arrival of a financing-investment opportunity. Note that when $\lambda = 0$ we are back to the baseline model with a single financing-investment opportunity.

Investment can be financed by issuing defaultable consols via competitive debt markets (as described in Section 2.3, but with prices reflecting the dynamic decisions of the firm) or equity. This implies that, in contrast to the model of Section 2, liabilities are no longer constant. Rather, $L(t)$ is now a pure jump process with $\mathrm{d}L(t) = (\hat{L}(t) - L(t^-))\mathrm{d}\mathbb{N}(t)$, where $\hat{L}(t)$—as defined in Section 2.1—denotes the value of liabilities immediately after investment implied by equity holders' investment and financing decisions.

4.2 Optimal Investment Problem

Conditional on the arrival of a financing-investment opportunity, the firm solves the natural analogue of the one-shot problem, except that both the $v(\cdot)$ and $p(\cdot)$ functions account for the possible arrival of future financing-investment opportunities. That is, upon arrival of an investment opportunity, the problem faced by equity holders is given by

$$v^*(\ell) = \max_{\substack{g, \hat{\ell} \geq 0 \\ \psi \in [0,1] \\ 0 \leq m \leq \kappa}} \left\{ (1+g)v(\hat{\ell}) - (1-\psi)q(g) + m \right\} \tag{30}$$

$$\text{s.t. budget constraint (23) and feasibility constraint (24)}$$

The following proposition describes the equity and debt holders' problems with repeated investment

Proposition 7 (Repeated Investment)**.** *A solution consists of a value of equity $v(\ell)$, price $p(\ell)$, policies $\{g(\ell), m(\ell), \psi(\ell), \hat{\ell}(\ell)\}$, and default threshold, $\bar{\ell}$, such that*

1. *Given $v(\ell)$ and $p(\ell)$:(a) the policies $\{g(\ell), m(\ell), \psi(\ell), \hat{\ell}(\ell)\}$ solve the firm's problem in* (30)*; (b) $v(\ell)$ satisfies the differential variational equation (DVI)*

$$0 = \min\{(r-\mu)v(\ell) + \mu\ell v'(\ell) - \frac{\sigma^2}{2}\ell^2 v''(\ell) - \lambda\left(v(\hat{\ell}(\ell)) - v(\ell)\right) - (1 - r\ell), v(\ell)\} \tag{31}$$

2. *The default threshold $\bar{\ell}$ is optimal and is determined by the indifference in* (31)

3. Given $v(\ell)$ and the equity holders' policies, the price $p(\ell)$ solves

$$rp(\ell) = r + (\sigma^2 - \mu)\ell p'(\ell) + \frac{\sigma^2}{2}\ell^2 p''(\ell) + \lambda\left(p(\hat{\ell}(\ell)) - p(\ell)\right) \tag{32}$$

$$p(\bar{\ell}) = \frac{(1-\theta)v(0)}{\bar{\ell}} \tag{33}$$

With boundary conditions $\lim_{\ell\to 0} v'(\ell) = 0$ and $\lim_{\ell\to\infty} v'(\ell) = 0$ for the firm's value function and $\lim_{\ell\to 0} p'(\ell) = 0$ for bond pricing to ensure the problem is well-posed. Furthermore, the first-best investment choice as defined in Definition 1 is

$$g^u = \frac{1}{\zeta(r-\mu)\left(\frac{1}{2}\left(\sqrt{1-\frac{2\lambda}{\zeta(r-\mu)^2}}-1\right)+1\right)} \tag{34}$$

Proof. See Appendix C. □

Unlike the one-shot case, we do not have closed-form solutions when $\lambda > 0$. Thus, we need to solve the model numerically using upwind finite difference methods. See Appendix D for a detailed description of the algorithm to solve for the Markov Perfect Equilibrium in Proposition 7. This equilibrium is not time-consistent in the same sense as Section 3.6. Equity holders solving (30) want to commit to new bond holders they will maintain a *future* investment policy of g^u—while still diluting existing bondholders claims to whatever extent possible by "double-selling" existing liabilities as claims in default to the new bondholders. However, limited-liability would make that commitment incredible.

4.3 Analysis

Figure 3 depicts investment relative to first-best ($\tilde{g} \equiv g/g^u$) and post-investment leverage relative to its pre-investment level ($\hat{\ell}/\ell$) plotted against leverage when $\lambda = 0.2$ (left panel) and $\lambda = 0.3$ (right panel), for different values of κ. It shows that the repeated arrival of financing-investment opportunities generates heterogeneous investment distortions, with low leverage firms tending to underinvest and high leverage firms tending to overinvest.[20] This heterogeneous effect of limited liability on equity holders' investment decisions is our key finding. Figure 3 also shows that firms may overinvest even if they are deleveraging. This observation further highlights that our mechanism differs from the mechanisms emphasized in earlier work on risk-shifting and debt dilution. We next discuss the main intuition behind equity holders' investment and leverage choices

[20]See Appendix F for detailed discussion of other parameters' values.

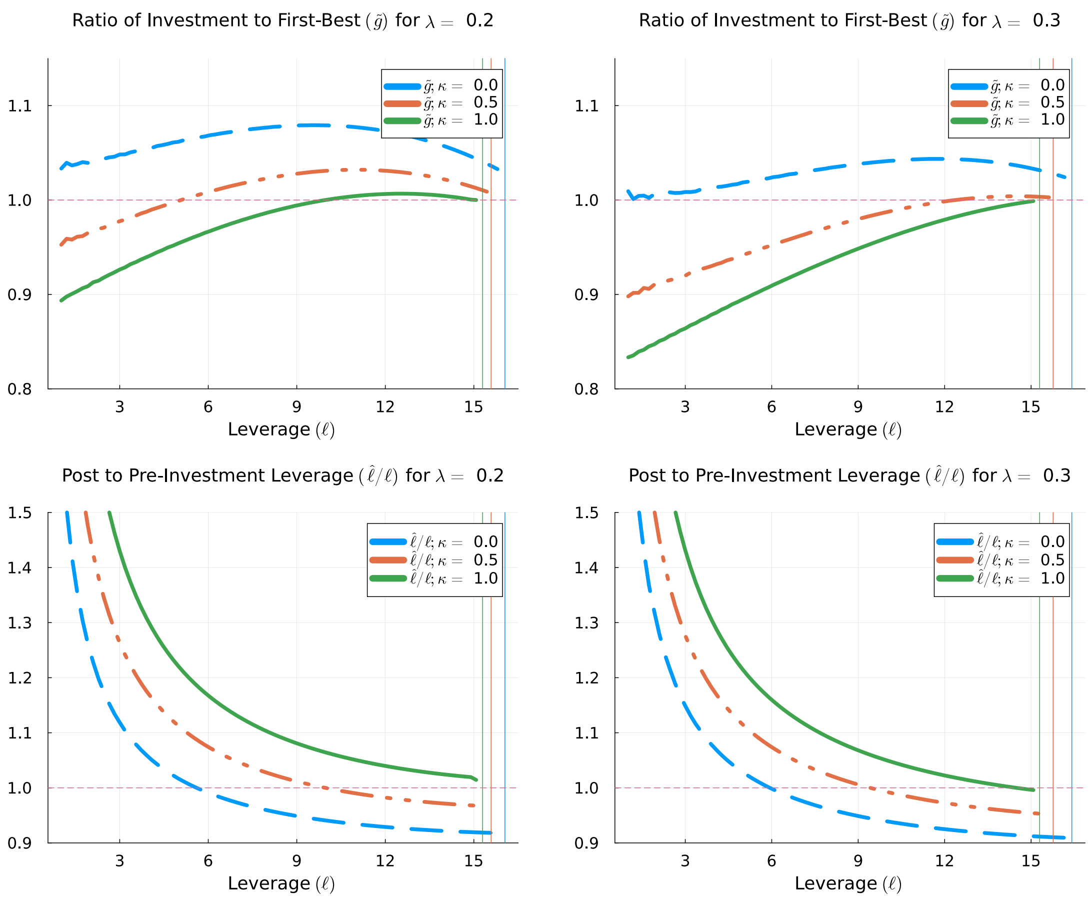

Figure 3: **The effect of equity payout restrictions on the investment-leverage relationship.** The top panels show investment relative to first-best ($\tilde{g}$) against preexisting leverage (ℓ), while the bottom panels show the ratio of ex-post to ex-ante leverage ($\hat{\ell}/\ell$). Each line corresponds to a different value for the constraint on equity payouts κ. The case $\kappa = 0$ is the baseline case of no equity payouts from debt. The left and right panels show this for an arrival rate of new financing-investment opportunities $\lambda = 0.2$ (left) and $\lambda = 0.3$ (right). Bankruptcy costs are set to zero ($\theta = 0$). Vertical lines indicate the default thresholds for each value of κ. Parameter values are discussed in Appendix F.

Equity holders' choice of investment Consider first the effect of an increase in the arrival rate of financing-investment opportunities, λ, on equity holders' investment decisions for a given level of κ. We see that a higher arrival of financing-investment opportunities decreases equity holders' investment relative to the first-best level for any level of κ. This is driven by two effects. First, a higher λ implies that equity holders have more opportunities to dilute existing debt holders, if such dilution is profitable. Therefore, new creditors expect their claims to be diluted sooner and require to be compensated for that, which increases the cost of debt financing. Second, an increase in λ indirectly increases the cost of inefficient investment.[21] The first of these forces is responsible for the large decrease in investment when κ is high and among low leverage firms when κ is low, while the second effect explains a reduction in investment among high-leverage firms when κ is low.

Consider next the effect of an increase in equity holders' ability to make equity payouts, κ, for a given level of λ. We see that an increase in κ decreases the extent of equity holders' overinvestment. Moreover, this decrease is disproportionally larger for firms with low leverage inducing these firms to underinvest. This is because an increase in κ decreases the cost of dilution as equity holders can now increase their leverage by increasing their equity payouts instead of engaging in inefficient investment. This effect is the strongest when ℓ is relatively low since low leverage firms have the largest capacity to increase leverage (and, hence, dilute debt holders). Since new creditors anticipate this behavior, the cost of debt increases sharply for low leverage firms', leading them to underinvest. As κ increases, this mechanism becomes relevant also for firms with higher levels of leverage, and when $\kappa = 1$ most firms underinvest for a wide range values of leverage, ℓ. Thus, the tightness of equity payouts constraints plays an important role in determining for which values of ℓ equity holders are prone to overinvesting.

Finally, note that for the highest levels of leverage, Figure 3 shows that investment converges to the first-best as κ increases, similarly to the one-shot model. This is because for high enough ℓ and κ equity holders are able to issue so much debt that they optimally choose to default immediately after investment. This implies that debt holders immediately take over the firm and equity holders have no opportunities to dilute new debt holders' claims. Thus, as long as there are no deadweight bankruptcy costs (or these costs are small), equity holders' and new debt holders' incentives are again aligned, just as in the model with one-shot investment.

Equity holders' choice of leverage Next, we consider equity holders' leverage policy given λ and κ. We see that low leverage firms increase their leverage more aggressively than high-leverage firms. This is because low leverage firms find it easier to increase their leverage: their initial leverage is low so even a small debt issuance increases their leverage above its current level. On the other hand, the same amount of debt issuance for high leverage firms leads to a much more modest change in their post-investment leverage. Since increasing leverage is costly (because at least part of it is driven by inefficient investment whose costs are fully borne by equity holders), firms with high leverage adjust their leverage less aggressively than low leverage.

[21] To see this note that the last part of Proposition 7 implies that the first-best unconstrained investment is an increasing function of λ. Since the investment cost is strictly convex and is borne by equity holders, overinvestment is more costly when λ is high.

Somewhat more surprisingly, we see that when equity payout constraints are strict (i.e., when κ is low), firms with high leverage choose to deleverage. This is driven by the fact that when inefficient investment is the main way to increase leverage the investment needed to increase leverage is an increasing function of ℓ.[22] Therefore, as ℓ increases, inefficient investment needed to increase leverage becomes exceedingly costly and, hence, firms with high ℓ choose to deleverage. However, note that this deleveraging is still associated with overinvestment. This is because, on the margin, increasing investment above its efficient level and financing it with debt allows the equity holders to limit the value of cash flows generated by new investment that are captured by existing debt holders.

Finally, we see that an increase in κ increases the post-investment leverage of all firms. An increase in λ has a more subtle effect, but it tends to decrease the leverage of high leverage firms. This decrease is driven by the fact that the price of debt for high leverage firms actually increases when λ increases since these firms deleverage when they invest, and investment is now more frequent. As such, when λ increases these firms can finance their choices with less debt.

Comparison with one-shot investment Two general observations arise when comparing the above results with the results derived in the model with one-shot investment. First, the tendency of equity holders' to overinvest extends to the model with repeated financing-investment opportunities, and the extent of overinvestment decreases with the arrival rate of new financing opportunities. Second, the model with repeated investment opportunities shows that allowing equity payouts financed with debt (i.e., $\kappa > 0$) need not improve the efficiency of firms' investment and may even induce equity holders to switch from overinvestment to underinvestment. Intuitively, when κ is high debt holders' concerns about the future dilution of their claims are exacerbated when equity holders can make direct payouts to themselves financed with debt more frequently.

Covenants In the model, for simplicity, we assumed that existing debt is not protected by covenants. In Appendix E, we argue that standard covenants such as restrictions on payouts, secure debt restrictions, restrictions on leverage, and senior debt restrictions do not resolve the time-inconsistency issue we highlight in our paper.

4.4 Dynamics of Leverage, Investment, and Bankruptcy

Figure 4 compares the simulated distribution of paths (including means and the interior 95% quantiles with 1000 simulations) of leverage and real investment of initial conditions (one with high leverage and a second one with lower leverage) across different equity payout

[22] See the discussion following Proposition 4.

restrictions. The left panels assume that equity payouts are prohibited ($\kappa = 0$), while the right panels permit equity buybacks ($\kappa = 0.5$).[23] Low-leverage firms (blue) are calibrated to have an initial interest coverage ratio of 4, consistent with Palomino et al. (2019) reporting an average interest coverage ratio of around 4 for the period 1970-2017. High-leverage firms (red) are calibrated to have an initial interest coverage ratio of 1.5, representing a typical below investment grade firm.[24] The top panels show simulated leverage and the bottom panels simulated real investment.

The top panels show that regardless of the value of κ, firms over time increase their leverage as predicted by the recent literature on leverage ratcheting (Admati et al. (2018) and DeMarzo and He (2021)). Comparing left and right panels we see that allowing equity payouts modestly increases the rate at which leverage increases. This increase is more pronounced for firms with low initial leverage as these firms have more capacity to take on more debt in order to finance direct payouts to equity holders. In addition, we see that increasing κ lowers the default threshold. This is driven by dynamic considerations, as new creditors price in future anticipated dilution, and thereby reduce equity holders' incentive to keep the firm as a going concern at any level of cash flows. While the changes in leverage and default thresholds from permitting equity payouts appear visually small, bankruptcy is a tail event and therefore is affected by these changes. In our simulations, the cumulative five-year bankruptcy rate for firms with initially high leverage increases when equity payouts are permitted (from 28.5% for the case of $\kappa = 0$ to 32.0% when $\kappa = 0.5$). The increase in bankruptcy rates arises through a combination of the incentive to lever up and the lower optimal leverage at which equity holders walk away from the firm, as captured by the lower default threshold.

Comparing the two bottom panels shows how equity payouts restrictions affects real investment for high and low leverage firms. In the left panel ($\kappa = 0$) both types of firms overinvest, while in the right panel ($\kappa = 0.5$) low-leverage firms tend to underinvest while high-leverage firms tend to overinvest. Therefore, we see that restricting equity payouts (decreasing κ) leads to a higher investment, which pushes firms' investment further from the first-best. Thus, like Myers (1977), we find that restricting equity payouts increases investment, however, in contrast to him, we find that such restrictions are likely not desirable as they push firms to overinvest. As such our results contribute to the recent push back against proposals to restrict equity payouts such as share buybacks. See Guest et al. (2023)

[23] Grullon and Michaely (2002) estimate an equal-weighted ratio of total of equity payouts (including dividends, repurchases, etc.) to earnings of around 0.5 in 2000. We take this as an upper bound for equity payouts in our model because in practice not all dividends/equity buybacks are financed through new debt issuance.

[24] Palomino et al. (2019) consider firms with an interest coverage ratio of 2 or less as firms at risk of default.

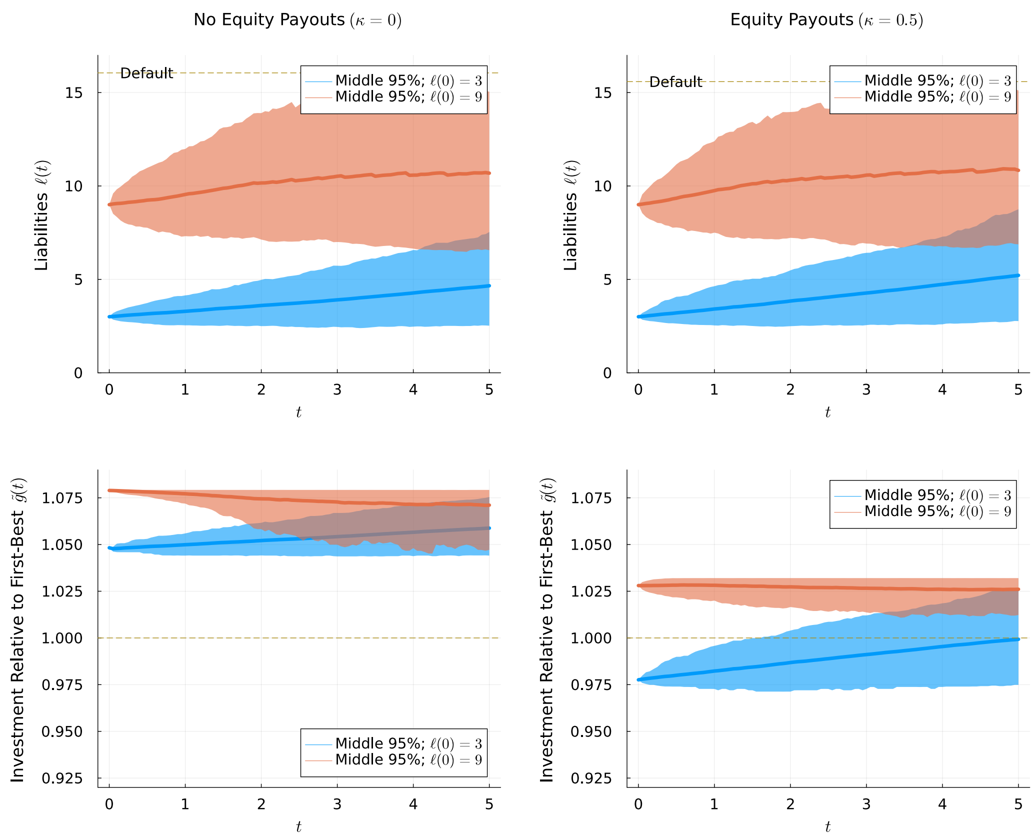

Figure 4: **Simulation of an ensemble of** 1000 **paths of leverage** ℓ **(top), investment relative to first-best** $\tilde{g}$ **(bottom) with and without equity payouts. blue=low initial leverage, red=high initial leverage.** Each panel shows the ensemble starting from $\ell(0) \in \{3, 9\}$ corresponding to interest coverage ratios of 4 and 1.5 respectively. The left panels use $\kappa = 0$ (no equity payouts) and the right panels use $\kappa = 0.5$ (constrained equity payouts from new debt). The arrival rate is set to $\lambda = 0.2$. The central line is the mean, the red shaded area shows the 2.5th and 97.5th percentiles. Moments are based on non-defaulted firms.

for an overview of arguments for and against such policies.

5 Discussion: Assumptions and Related Literature

In this section, we discuss (i) which assumptions are key for our results and (ii) relate our results to the existing literature.

Assumptions Three assumptions are key for overinvestment to emerge in our model: (1) deadweight costs of default are not "too large," (2) investment and financing opportunities arrive infrequently, and (3) new creditors capture high enough fraction of the firm's value in default.[25] We now discuss the role of each of them in more detail.

To understand why these assumptions matter note that when bankruptcy costs are high then the cost of issuing debt are also high. This is because the new creditors receive little compensation in the case of default are are willing to purchase debt only at low price. At the same time, high bankruptcy costs imply that the "double-selling" mechanism is weak. While issuing new debt continues to dilute the existing debt holders' coupon claims by bringing (in expectations) default time forward, few of these coupon claims are transformed into bankruptcy claims since most of the cash flows are destroyed in default. In the extreme case where recovery value is zero (as in DeMarzo and He (2021)) the "double selling" mechanism is absent. Together this implies that if a high fraction of firm's value is destroyed in default then equity holders underinvest.[26]

For the equity holders to benefit from investment, it also has to be the case that new creditors receive a high enough fraction of the firm in default. To understand why this is the case, suppose that new debt holders do not have any claims to the value of firm in default (receive nothing in the case of default). Then, while overinvestment dilutes the existing debt holders' coupon claims transforming them into bankruptcy claims, equity holders cannot benefit from it since all those new bankruptcy claims are captured by existing debt holders (are not double-sold to new creditors). Thus, in order for overinvestment to benefit equity holders, it has to be the case that new creditors capture enough of the value of the firm in

[25] As stressed above, whether overinvestment occurs in equilibrium also depends on the tightness of equity payouts restrictions. DeMarzo and He (2021) and the literature on leverage dynamics typically do not impose such restrictions. On the other hand, investment literature commonly assumes that either the amount of outstanding debt is fixed (see, for example, Hennessy (2004) or Diamond and He (2014)) or new debt is only issued to finance investment (see, for example, Hackbarth and Mauer (2012)). Thus, our model falls in between these two extremes. Our results also emphasize the importance of this assumption, which is typically made without much discussion even though covenants restricting equity payouts and subsequent debt issuance are common, particularly among bonds issued by below-investment grade and unrated firms.

[26] If debt financing cost are so high that equity holders prefer to use equity financing then we are back to the standard debt overhang case as in Myers (1977), where equity holders underinvest since part of the value of new investment is captured by existing debt holders.

default (as new debt holders pay equity holders the fair value for all the cash flows they will receive in the future). A sufficient condition for this to occur is that all debt is unsecured and has equal priority in default (as we assume in the benchmark model).[27]

Finally, in the setup with repeated investment, equity holders are more likely to overinvest if investment opportunities arrive relatively infrequently. This is because, new creditors expect equity holders to dilute their claims in the future by issuing additional debt. The higher the arrival rate the sooner creditors expect to be diluted and, hence, the lower is the price of new debt. The resulting increase in the cost of financing investment dominates the benefit from diluting existing debt holders leading to underinvestment. On the other hand, when arrival rate is low, the benefit from dilution dominate and equity holders overinvest.

Relation to existing literature It is useful to contrast our findings with existing literature on debt overhang in dynamic models based on Leland (1994). This literature emphasized that in the presence of existing debt firms' underinvest, as predicted by Myers (1977) (see Hennessy (2004) or Diamond and He (2014) and references therein). In contrast, to that earlier literature which focused on equity financing, we allow equity holders to choose their financing optimally using combination of equity and debt. Closely related is DeMarzo and He (2021) who also consider optimal financing of investment. Considering a similar framework to ours, they find that equity holders always underinvest. The reason for our contrasting conclusions is that DeMarzo and He (2021) assume that (i) bankruptcy results in a complete loss of the firm's value all default and (ii) investment decisions are made continuously. In contrast, we consider the case where firm's value is (mostly) preserved in default and investment opportunities arrive infrequently. As explained above, these assumption imply that in our model equity holders tend to have an incentive to overinvest.

It is also worth comparing our results with models of financial frictions that feature collateral constraints. These models also assume that firms are protected by limited liability, but they also feature additional sources of market incompleteness such as private information (e.g., Bernanke et al. (1999) or Clementi and Hopenhayn (2006)) or ability to abscond funds (e.g., Buera et al. (2011) or Moll (2014)). In those models, the ability to issue equity payouts would have no effect on constrained firms' investment choices since, in those models, it is typically optimal to delay dividends (see, for example, Clementi and Hopenhayn (2006)). In addition, a higher arrival of financing-investment opportunities would either have no effect (if collateral constraints are modeled as in Buera et al. (2011) or Moll (2014)) or would lead to an increase in investment relative to first-best due to an implied increase in firms' future profitability (if collateral constraints are modeled as in

[27] Alternatively, this happens when all existing debt is collateralized with existing assets (see Section 2.3).

Clementi and Hopenhayn (2006)). In contrast, we find that equity payouts tend to decrease firms' investment relative to first-best, resulting in underinvestment by low leverage firms but limiting overinvestment by high-leverage firms. Furthermore, in our model, a higher arrival rate leads to a downward shift of investment policy resulting in less investment for any given value of leverage.

6 Conclusion

This paper provides a dynamic model showing that limited liability can inefficiently distort investment away from low-leverage firms and towards highly leveraged firms. In our model high leverage firms have an incentive to overinvest because debt financed investment allows them to (1) dilute current debt holders' coupon claims by increasing leverage and bringing forward bankruptcy, while capitalizing on this by "double-selling" default claims; or (2) limit the gains from investment to debt holders from new investment if increasing leverage is too costly. At the same time, our model predicts that low leverage firms with frequent investment-financing opportunities tend to underinvest. This is because these firms have the largest capacity to dilute the coupons of debt holders in the future, which is anticipated by creditors who require high compensation for lending to those firms. These mechanisms continues to hold when all debt is collateralized and strengthen when equity holders can deplete equity by making equity payouts from new debt issuances.

Heterogeneous investment distortions have potentially important consequences for misallocation of resources in the economy and through it on aggregate productivity and welfare. In particular, our results that highly leveraged firms overinvest while firms low leverage underinvest leads to both within and between sector misallocation. Within sector, our model implies that resources will be skewed towards highly leveraged firms, which tends to be larger (Rajan and Zingales (1995) or Chatterjee and Eyigungor (2023)). This potentially might leads to greater concentration of sales, less competition, and inefficiently low entry of new firms. Across sectors, this can lead to misallocation of resources towards sectors with high leverage such as oil and extraction away from sectors with low leverage such as pharmaceuticals (see Myers (2001)).

Our work has also several policy implications. One lesson from our analysis is that restrictions on equity payouts are no cure-all to excessive leverage and low investment, and the effects of such a policy are heterogeneous by initial firm leverage. In our model, equity payout restrictions reduce bankruptcy and may raise investment towards the first-best when initial leverage is low. However, equity payout restrictions tend to exacerbate inefficient overinvestment when initial firm leverage is high. Our analysis also suggests that investment subsidies targeted at small firms that tend to be less leveraged and, hence more prone to underinvestment, might be a good tool for decreasing misallocation of resources and bringing aggregate investment closer to its efficient values. Related idea has been recently explored by Cuciniello et al. (2023) who study efficient business start-up subsidies in general equilibrium framework that captures main trade-offs explored in DeMarzo and He (2021) and our paper.

To emphasize the role of limited liability, we abstracted from other related mechanism and focused on decisions of a single firm. The model can be extended to allow for information frictions in the quality of collateral (Gorton and Ordoñez (2014)). If collateral quality is cyclical, debt financing would likely be further distorted relative to our benchmark. The channel emphasized in this paper likely has potentially important aggregate implications, both through investment distortions across firms (e.g., Khan and Thomas (2013), Moll (2014), and Buera et al. (2011)), and the cyclicality of the costs of financial distress (e.g. Atkeson et al. (2017)). We believe that investigating these general equilibrium consequences of limited liability will be fruitful.

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Appendix A Proofs for Section 2

A.1 Proof of Proposition 1

Proof of Proposition 1. To find the post-investment value of equity $V(Z,L)$ that solves equity holders' default problem as described by (6)- (8) we use the method of undetermined coefficients with the guess

$$V(Z,L) = \frac{1}{r-\mu}\left(Z + \frac{\omega}{\eta}Z^{-\eta}\right) - L \tag{A.1}$$

Using this guess in (6) and equating undetermined coefficients we arrive at the equation $2(r+\eta\mu) = \eta(1+\eta)\sigma^2$. We solve this quadratic equation for η and note that the smaller of the two roots is explosive and, hence, it violates the transversality condition. Therefore,

$$\eta = \frac{\left(\mu - \sigma^2/2\right) + \sqrt{\left(\mu - \sigma^2/2\right)^2 + 2\sigma^2 r}}{\sigma^2} \tag{A.2}$$

Next, we substitute the guess (A.1) into (8) to find that $\omega = \underline{Z}^{\eta+1}$. Then we use (A.1) and the expression for ω in (7) to find that the default threshold is given by

$$\underline{Z} = \frac{(r-\mu)\eta}{\eta+1}L \tag{A.3}$$

(A.3) defines the default threshold and completes derivations of $V(Z,L)$. Since the default threshold depends on equity holders' liabilities L we donate it by $\underline{Z}(L)$.

Next, we show that the value of equity can be expressed as $V(Z,L) = v(\ell)Z$, where $\ell \equiv L/Z$, and derive the expression for $v(\cdot)$. First, we note that

$$\frac{\underline{Z}(L)}{Z} = \frac{(r-\mu)\eta}{\eta+1}\frac{L}{Z} = \frac{(r-\mu)\eta}{\eta+1}\ell, \tag{A.4}$$

Next, we use the expressions for ω and for $\underline{Z}/Z$ found in (A.1) to obtain

$$V(Z,L)/Z = \frac{1}{r-\mu} + \frac{\chi}{\eta+1}\ell^{\eta+1} - \ell, \tag{A.5}$$

where

$$\chi \equiv \left(\frac{(r-\mu)\eta}{\eta+1}\right)^{\eta} \tag{A.6}$$

Finally, we set $v(\ell) = \frac{1}{r-\mu} - \ell(1 - s(\ell))$, were $s(\ell) = \frac{\chi}{\eta+1}\ell^{\eta}$, which implies that $V(Z,L)/Z = v(\ell)$.

To find the liquidation value of the firm we note that equity holders walk away when cash flows Z reach the default threshold $\underline{Z}(L)$. At that time debt holders take over the firm so its liabilities are reset to $L = 0$ but the firm loses fraction $\theta \in [0,1]$ of its value. It follows that the liquidation value of the firm, from creditors' perspective, is given by

$$V((1-\theta)\underline{Z}(L),0) = \frac{(1-\theta)\underline{Z}(L)}{r-\mu} \tag{A.7}$$

Thus, the liquidation value per unit of liabilities is given by

$$\frac{V((1-\theta)\underline{Z}(L),0)}{L} = \frac{(1-\theta)\eta}{\eta+1}, \tag{A.8}$$

where we used the definition of $\underline{Z}(L)$ (see (A.3).

□

A.2 Proof of Proposition 2

Proof of Proposition 2. **(Derivations of debt prices)** Let T be the first-time cash flows, Z, reach the default threshold $\underline{Z}(L)$. Since Z follows a geometric Brownian motion (see (1)) we have

$$\mathbb{E}_T\left[e^{-rT}\right] = \exp\left(\frac{-(\mu-\sigma^2/2)-\sqrt{(\mu-\sigma^2/2)^2+2\sigma^2 r}}{\sigma^2}(\log Z - \log \underline{Z}(L))\right) \tag{A.9}$$

as shown in Jeanblanc et al. (2009). Using the definition of η and χ (see (A.2) and (A.6), respectively) and the expression obtained in (A.3) we conclude that

$$\mathbb{E}_T\left[e^{-rT}\right] = (Z/\underline{Z}(L))^{-\eta} = \chi\ell^{\eta} \tag{A.10}$$

Using (A.10) we see that

$$P^C(Z,L) \equiv \frac{p^C(Z,L)}{r} = \frac{1}{r}\left[1-\chi\ell^{\eta}\right] = \frac{1}{r}\left[1-(1+\eta)s(\ell)\right], \tag{A.11}$$

where $s(\ell) = \chi/(1+\eta)\ell^{\eta}$ and $\ell = L/Z$. Note that the above equation implies that the relevant state variable is ℓ. Hence, the price of coupon claims depends only in ℓ, and we can denote that price by $P^C(\ell)$.

Next, we consider the price of a bankruptcy claim $P^B(Z,L)$. Note that

$$P^B(Z,L) \equiv \frac{p^B(Z,L)}{r} = \frac{V((1-\theta)\underline{Z}(L),0)}{rL}\mathbb{E}_T\left[e^{-rT}\right] = \frac{1}{r}(1-\theta)\eta s(\ell), \tag{A.12}$$

where we used (A.8), (A.9), and the definition of $s(\ell)$. We see again that the relevant state variable is ℓ and, thus, we can write the price of claims bankruptcy as $P^B(\ell) \equiv \frac{p^B(\ell)}{r}$.

From the above discussion it follows that the leverage ℓ is the relevant state for pricing defaultable consols so that we can write $P(Z,L) = P(\ell)$ and $p(Z,L) = p(\ell)$. Putting together (A.11) and (A.12) we obtain

$$P(\ell) = \frac{p(\ell)}{r} = \frac{1}{r}\left[1-(1+\theta\eta)s(\ell)\right] \tag{A.13}$$

(The Budget Constraint) Equity holders issue debt to finance their equity payouts, M, and a fraction ψ of the investment cost $Zq(g)$. Let K denote the quantity of new bonds issued by equity holders to finance $\psi Zq(g) + M$. Then, K has to satisfy the following budget constraint

$$P(\hat{\ell})K = \psi Zq(g) + M, \tag{A.14}$$

where $\hat{\ell}$ is the post-investment leverage. Next, we relate K to the change in leverage $\hat{L} - L$. Recall that L is defined as the present discounted value (PDV) of liabilities. Since each unit of debt promises a payment of a constant coupon of 1 and agents discount these payments at a rate r it follows the PDV of the cash flows promised to new debt holders is given by K/r. Therefore, the post-investment liabilities are given by $\hat{L} = L + \frac{K}{r}$. It follows that $K = r(\hat{L} - L)$. Substituting this expression for K into the budget constraint (A.14), dividing both sides of the resulting equation by Z, and using the definition of $p(\cdot)$ (see (A.13)) we obtain

$$p(\hat{\ell})\left(\hat{\ell}(1+g) - \ell\right) = \psi q(g) + m, \tag{A.15}$$

where $\hat{\ell} = \hat{L}/(Z(1+g))$. This corresponds to (20) in the text. □

(Collateralized Debt) We formally show that (i) the price of debt where default claims are pledged as collateral (as opposed to the proportional claims to the firm in bankruptcy) is identical to the price of debt in our baseline model under suitable pledgeability constraint and (ii) under pledgeability constraints, equity holders choices respect constraint (21) that we impose in the benchmark model.

To consider collateralized debt we interpret the value of the firm in bankruptcy, $\frac{\eta}{1+\eta}L$, as pledgeable collateral.[28,29] Let C denote the collateral promised to the existing debt holders. We assume that the firm used all of its assets in available in bankruptcy as collateral when it issued existing debt in the past —as would be optimal— so that $C = \frac{\eta}{1+\eta}L$. In order for the equity holder to deliver this value following their investment and financing choices, it has to be the case that

$$\frac{\eta}{1+\eta}\hat{L} \geq \frac{\eta}{1+\eta}L \tag{A.16}$$

and that only

$$\min\left\{\frac{\hat{Z}}{r-\mu}, \frac{\eta}{1+\eta}\hat{L}\right\} - \frac{\eta}{1+\eta}L \tag{A.17}$$

can be used for collateral for the new debt.[30] We refer to (A.16) and (A.17) as the pledgeability

[28] This interpretation is possible since the value of the firm in bankruptcy is known with certainty since investment is deterministic and the post-investment paths for $Z(t)$ are continuous.

[29] As in the benchmark model, equity holders' liability are equal to the PDV of promised coupon payments. For that reason, equity holders' liabilities are unchanged if debt is collateralized or carry a proportional claim to the value of the firm in default.

[30] The maximum value that equity holders can possibly promise as collateral is the post-investment

constraints.

Now, consider the price of new collateralized debt, denoted by $\hat{P}^S(\hat{Z}, \hat{L}, \hat{C})$, where $\hat{Z}$ and $\hat{L}$ are post-investment cash flows and liabilities, respectively, and $\hat{C}$ is the value of collateral promised to new debt holders. Then, following the same argument as we used in the proof of Proposition 2, we obtain

$$\hat{P}^S(\hat{Z}, \hat{L}, \hat{C}) = P^C(L, Z) + \frac{\hat{C}}{r(\hat{L} - L)}\chi\left(\frac{\hat{L}}{\hat{Z}}\right)^{\eta}, \tag{A.18}$$

where $P^C(L, Z)$ is defined in (16). Since the value of new debt is increasing in collateral promised and equity holders do not get to keep any collateral unused, it follows that $\hat{C} = \min\left\{\frac{\hat{Z}}{r-\mu}, \frac{\eta}{1+\eta}\hat{L}\right\} - \frac{\eta}{1+\eta}L$. Therefore, if $\frac{\eta}{1+\eta}\hat{L} \leq \frac{\hat{Z}}{r-\mu}$ (as we argue below), then

$$\hat{P}^S(\hat{Z}, \hat{L}, \hat{C}) = P^C(\hat{L}, \hat{Z}) + \frac{1}{r}\frac{\eta}{1+\eta}\chi\left(\frac{\hat{L}}{\hat{Z}}\right)^{\eta} = P^C(\hat{L}, \hat{Z}) + P^B(\hat{L}, \hat{Z}) = P(\hat{L}, \hat{Z}) \tag{A.19}$$

This implies that the price of new debt is exactly the same as in the benchmark model. A similar argument can be used to show that the price of existing collateralized debt, which we denote by $P^S(Z, L, C)$, is equal to $P(Z, L)$.

We now argue that equity holders find it optimal to choose $\hat{L} \leq \frac{\hat{Z}}{r-\mu}$. To see that this is the case, note that choosing $\hat{L} > \frac{\hat{Z}}{r-\mu}$ yields the same payoff as $\hat{L} = \frac{\hat{Z}}{r-\mu}$. This is because in both cases equity holders decide to walk away from the firm and the total value of the new debt issued is the same. From the above discussion, we know that if $\hat{L} \leq \frac{\hat{Z}}{r-\mu}$ then equity holders' problem is the same as in the baseline model. Thus, if $\kappa = 0$ or $\kappa < \underline{\kappa}$ (where $\underline{\kappa}$ is defined as in Proposition 5) then equity holders strictly prefer to choose $\hat{L} \leq \frac{\hat{Z}}{r-\mu}$. If $\kappa \geq \underline{\kappa}$ then equity holders are indifferent between any $\hat{L} \geq \frac{\hat{Z}}{r-\mu}$ and so we can assume that they choose $\hat{L} = \frac{\hat{Z}}{r-\mu}$. This completes our argument.

A.3 Proof of Proposition 3

Proof of Proposition 3. To derive (22) consider equity holders' objective function (30) and substitute the budget constraint ((A.15)) to eliminate $\psi q(g) + m$ and obtain

$$(1+g)v(\hat{\ell}) - q(g) + p(\hat{\ell})(\hat{\ell}(1+g) - \ell) \tag{A.20}$$

Using the expression we found for $v(\hat{\ell})$ (Proposition 1), (A.20), the observation $p(\hat{\ell}) = 1-(1+\theta\eta)s(\hat{\ell})$, and simplifying we obtain

$$\frac{1+g}{r-\mu} - p(\hat{\ell})\ell - q(g) - \theta\eta s(\hat{\ell})\hat{\ell}(1+g) \tag{A.21}$$

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value of the firm (if they choose to default immediately), $\frac{\hat{Z}}{r-\mu}$, net of the collateral promised to existing debt holders. This happens if equity holders choose $\frac{\eta}{1+\eta}\hat{L} > \frac{\hat{Z}}{r-\mu}$. Otherwise, if equity holders choose optimally to continue operating the firm, the maximal value of collateral they can promised to new debt holders is $\frac{\eta}{1+\eta}(\hat{L} - L)$.

Defining $H(\hat{\ell}) = \theta\eta s(\hat{\ell})$ and using this definition in (A.21) we obtain the equity holders' objective function (22) in Proposition 3.

To obtain (25), note that $v(0) = \frac{1}{r-\mu}$. Thus, (15) implies that g^u is a unique solution to the F.O.C. given by $0 = \frac{1}{r-\mu} - q'(g^u)$. □

Appendix B Proofs for Section 3

In this section, we provide proofs of propositions stated in Section 3 (Propositions 4 and 5). We begin by establishing a number of useful preliminary results and by discussing feasibility of equity holders' choice of $\{g, \hat{\ell}, m, \psi\}$ (Appendix B.1).

B.1 Preliminary Results

We establish first a number of preliminary results that we will use to prove Propositions 4 and 5. Readers primarily interested in main results may choose to skip this section.

Lemma 1. *Define $\bar{\ell}$ as the unique solution to*

$$1 = \chi\bar{\ell}^{\eta}, \tag{B.1}$$

Then $p(\bar{\ell})\bar{\ell} = \frac{1}{r-\mu}$.

Proof. Plugging $\bar{\ell}$ into the expression for $p(\ell)$ (see (17)) we obtain $p(\bar{\ell}) = \frac{\eta}{1+\eta}$. From the definitions of $\bar{\ell}$ and χ ((B.1) and (10), respectively) we obtain

$$\bar{\ell} = \frac{1+\eta}{\eta}\frac{1}{r-\mu} \tag{B.2}$$

Combining these observations we obtain $p(\bar{\ell})\bar{\ell} = \frac{1}{r-\mu}$. □

In the proof of Lemma 1 we derived expressions for $\bar{\ell}$ and $p(\bar{\ell})$. Since we make use of these expressions repeatedly, we report them in the following Corollary.

Corollary 2. *We have $\bar{\ell} = \frac{1}{r-\mu}\frac{1+\eta}{\eta}$ and $p(\bar{\ell}) = \frac{\eta}{1+\eta}$*

Next, we show the constraint $p(\ell) \geq p^B(\ell)$ is satisfied if and only if $\ell \in [0, \bar{\ell}]$.

Corollary 3. *We have $p(\ell) \geq p^B(\ell)$ if and only if $\ell \in [0, \bar{\ell}]$.*

Proof. From the definitions of $p(\ell)$ and $p^B(\ell)$ (see Proposition 2) we have $p(\ell) - p^B(\ell) = 1 - \chi\ell^{\eta}$. Since $\chi > 0$ and $\eta > 0$ it follows that $p(\ell) - p^B(\ell)$ is strictly decreasing in ℓ. Moreover, from the definition of $\bar{\ell}$ we see that $p(\bar{\ell}) - p^B(\bar{\ell}) = 0$. This establishes the claim. □

Having characterized the highest feasible leverage, $\bar{\ell}$, we now establish two useful results regarding the value of outstanding debt.

Lemma 4. *For all $\hat{\ell} < \bar{\ell}$ we have*

$$\frac{\partial}{\partial \hat{\ell}}\left[p(\hat{\ell})\left(\hat{\ell}(1+g)-\ell\right)\right] = p'(\hat{\ell})\left(\hat{\ell}(1+g)-\ell\right) + p(\hat{\ell})(1+g) > 0 \tag{B.3}$$

Proof. We have

$$\frac{\partial}{\partial \hat{\ell}}\left[p(\hat{\ell})\left(\hat{\ell}(1+g)-\ell\right)\right] = \left[1-\chi\hat{\ell}^{\eta}\right](1+g) + \frac{\eta\chi}{1+\eta}\hat{\ell}^{\eta-1}\ell, \tag{B.4}$$

where we used the definition of $p(\ell)$ (see (17) with $\theta = 0$). The claim follows from the observation that, since $\hat{\ell} < \bar{\ell}$, we have $\left[1-\chi\hat{\ell}^{\eta}\right] > 0$. □

Lemma 5. *The value of outstanding debt, $p(\ell)\ell$, is strictly increasing in ℓ for all $\ell \in [0, \bar{\ell})$.*

Proof. We have

$$\frac{\partial}{\partial \ell} p(\ell)\ell = p'(\ell)\ell + p(\ell) = 1 - \chi\ell^{\eta} > 0, \tag{B.5}$$

where the last inequality follows since $\ell < \bar{\ell}$. □

Finally, we discuss which equity holders' choices of $g, \psi, m,$ and $\hat{\ell}$ are feasible (i.e., satisfy equity holders' budget constraint). Note that the equity holders' choices of $g, \psi, m,$ and $\hat{\ell}$ have to jointly satisfy the equity holders' budget constraint

$$p(\hat{\ell})((1+g)\hat{\ell} - \ell) = \psi q(g) + m \tag{B.6}$$

The budget constraint implies that once the equity holders make choices of $g, \psi,$ and m the post-investment leverage $\hat{\ell}$ is determined implicitly by (B.6). Thus, we can treat the post-investment leverage as an implicit function of $g, \psi,$ and m, and denote it by $\hat{\ell}(g, \psi, m)$. This leads to the following definition of feasibility of equity holders choices.

Definition 2. *The equity holders choices of g, ψ, m are feasible if $\hat{\ell}(g, \psi, m) \leq \bar{\ell}$*

We now derive a feasibility constraint on issuance of m.

Lemma 6. *Fix g and ψ such that $\hat{\ell}(g, \psi, 0) < \bar{\ell}$. Then m is feasible if $m \in [0, \bar{m}(g, \psi)]$, where*

$$\bar{m}(g, \psi) = \frac{1+g}{r-\mu} - \frac{\eta}{1+\eta}\ell - \psi q(g) \tag{B.7}$$

Moreover, at $m = \bar{m}(g, \psi)$ we have $\hat{\ell}(g, \psi, m) = \bar{\ell}$.

Proof. Since ψ and g are such that $\hat{\ell}(g, \psi, 0) < \bar{\ell}$ then, given choices of g and ψ there exists $m > 0$ that satisfies

$$p(\hat{\ell})\left(\hat{\ell}(1+g) - \ell\right) = \psi q(g) + m \tag{B.8}$$

By applying the implicit function theorem to the above equation we see that

$$\frac{\partial \hat{\ell}}{\partial m} = \frac{1}{p'(\hat{\ell})\left(\hat{\ell}(1+g) - \ell\right) + p(\hat{\ell})(1+g)} > 0, \tag{B.9}$$

where the inequality follows from Lemma 4. It follows that at the highest feasible m, which we denote by $\bar{m}(g, \psi)$, we have $\hat{\ell} = \bar{\ell}$. Setting $\hat{\ell} = \bar{\ell}$ in (B.8) and rearranging, we obtain

$$\bar{m}(g, \psi) = \frac{1+g}{r-\mu} - \frac{\eta}{1+\eta}\ell - \psi q(g)$$

□

B.2 Proof of Proposition 4 ($\kappa = 0$)

Proof of Proposition 4. **(Part 1)** Equity financing implies that $\psi = 0$. Therefore, the post-investment leverage $\hat{\ell}$ is given by $\hat{\ell} = \frac{\ell}{1+g}$ and equity holders' problem simplifies to

$$\max_{g \geq 0} \frac{1+g}{r-\mu} - p\left(\frac{\ell}{1+g}\right)\ell - q(g)$$

The first-order condition associated with the above problem is given by

$$\frac{1}{r-\mu} + p'\left(\frac{\ell}{1+g}\right)\left(\frac{\ell}{1+g}\right)^2 - q'(g) = 0 \tag{B.10}$$

Recall that g^u denotes the first-best investment (see Definition 1) and let g_e^* denote equity holders' optimal investment under equity financing. Then, (25) and (B.10) imply that

$$\frac{1}{r-\mu} - q'(g^u) = 0 = \frac{1}{r-\mu} + p'\left(\frac{\ell}{1+g_e^*}\right)\left(\frac{\ell}{1+g_e^*}\right)^2 - q'(g_e^*) < \frac{1}{r-\mu} - q'(g_e^*), \tag{B.11}$$

where the inequality follows from the observation that $p'(\ell) < 0$ for all ℓ. Since the cost function q is strictly increasing in g, (B.11) implies that $g_e^* < g^u$.

(Part 2) We now allow the equity holders to choose their financing of investment optimally so that $\psi \in [0, 1]$. In this case, the equity holders' problem is given by

$$\max_{\substack{g, \hat{\ell} \geq 0 \\ \psi \in [0,1]}} \quad \frac{1+g}{r-\mu} - p(\hat{\ell})\ell - q(g) \tag{B.12}$$

$$\text{s.t.} \quad p(\hat{\ell})\left(\hat{\ell}(1+g) - \ell\right) = \psi q(g) \tag{B.13}$$

$$p(\hat{\ell}) \geq p^B(\hat{\ell}) \tag{B.14}$$

where $m = 0$ since $\kappa = 0$. Recall that $\hat{\ell}(g, \psi, m)$ denotes the level of post-investment leverage implied by equity holders' choices via the budget constraint. Since $m = 0$ in what follows we slightly abuse notation and write $\hat{\ell}(g, \psi)$ instead to $\hat{\ell}(g, \psi, 0)$.

It is easy to see that, as long as $\ell < \bar{\ell}$ the equity holders will never choose g, ψ such that $\hat{\ell}(g,\psi) = \bar{\ell}$. This is because when $\hat{\ell}(g,\psi) = \bar{\ell}$ then equity holders' post-investment value of equity is 0 (equity holders immediately default) while $\ell < \bar{\ell}$ implies that the pre-investment value of equity is strictly positive. It follows that the constraint (B.14) (which, as shown in Corollary 3 is equivalent to the constraint $\hat{\ell} \leq \bar{\ell}$) is not binding. Hence, the equity holders' problem can be written as

$$\max_{\substack{g\geq 0\\ \psi\in[0,1]}} \quad \frac{1+g}{r-\mu} - p\left(\hat{\ell}(g,\psi)\right)\ell - q(g), \tag{B.15}$$

subject to $\psi \in [0,1]$, where $\hat{\ell}(g,\psi)$ is implicitly defined by (B.13). Note that the first-order derivative of equity holders' objective function (B.15) w.r.t. ψ is given by

$$-p'(\hat{\ell})\ell\frac{\partial\hat{\ell}}{\partial\psi} > 0 \tag{B.16}$$

since $p'(\hat{\ell}) < 0$ and $\partial\hat{\ell}/\partial\psi$ (obtained by applying the implicit function theorem to (B.13)) is given by

$$\frac{\partial\hat{\ell}}{\partial\psi} = \frac{q(g)}{p'(\hat{\ell})\left(\hat{\ell}(1+g)-\ell\right) + p(\hat{\ell})(1+g)} > 0 \tag{B.17}$$

It follows that equity holders find it optimal to finance all of their investment with debt, that is, $\psi^* = 1$.

Consider next the optimal choice of g. The optimal choice of g, which we denote by g^*, satisfies the following first-order condition

$$\frac{1}{r-\mu} - p'\left(\hat{\ell}\right)\ell\frac{\partial\hat{\ell}}{\partial g}\Big|_{\substack{\psi=1\\ g=g^*}} - q'(g^*) = 0, \tag{B.18}$$

where

$$\frac{\partial\hat{\ell}}{\partial g} = -\frac{p(\hat{\ell})\hat{\ell} - \psi q'(g)}{p'(\hat{\ell})(\hat{\ell}(1+g)-\ell) + p(\hat{\ell})(1+g)} \tag{B.19}$$

We now argue that $\partial\hat{\ell}/\partial g$ evaluated at $g = g^*, \psi = \psi^* = 1$ is strictly positive. To see this, note that (B.18) implies that

$$-q'(g^*) = -\frac{1}{r-\mu} + p'(\hat{\ell})\ell\frac{\partial\hat{\ell}}{\partial g}\Big|_{\substack{\psi=1\\ g=g^*}}$$

Using the above expression in (B.19) evaluated at $g = g^*, \psi = 1$ and rearranging, we obtain

$$\frac{\partial\hat{\ell}}{\partial g}\Big|_{\substack{\psi=1\\ g=g^*}}\left[\frac{(1+g^*)\left(p(\hat{\ell}) + p'(\hat{\ell})\hat{\ell}\right)}{p'(\hat{\ell})(\hat{\ell}(1+g^*)-\ell) + p(\hat{\ell})(1+g^*)}\right] = \frac{-p(\hat{\ell})\hat{\ell} + \frac{1}{r-\mu}}{p'(\hat{\ell})(\hat{\ell}(1+g^*)-\ell) + p(\hat{\ell})(1+g^*)} \tag{B.20}$$

From Lemma 1 and Lemma 5 we know that $-p(\hat{\ell})\hat{\ell} + \frac{1}{r-\mu} \geq 0$ with a strict inequality if $\hat{\ell} < \bar{\ell}$. However, as we argued above, choosing $\hat{\ell} = \bar{\ell}$ is not optimal. Thus, at optimal choices of investment and financing we have $-p(\hat{\ell})\hat{\ell} + \frac{1}{r-\mu} > 0$. Next, note that by Lemma 4 the denominator on the RHS of (B.20) is strictly positive. Thus, it follows that the RHS of (B.20) is strictly positive. Furthermore, (B.20). Lemmas 4 and 5 imply that the expression in square brackets on the LHS of (B.20) is strictly positive. Therefore, we conclude that

$$\frac{\partial \hat{\ell}}{\partial g}\Big|_{\substack{\psi=1 \\ g=g^*}} > 0 \tag{B.21}$$

Having established that $\partial\hat{\ell}/\partial g|_{\{g=g^*,\psi=1\}} > 0$ we consider again (B.18). Since, $p'(\hat{\ell}) < 0$ we have

$$0 = \frac{1}{r-\mu} - q'(g^*) - p'\left(\hat{\ell}\right)\ell \frac{\partial \hat{\ell}}{\partial g}\Big|_{\substack{\psi=1 \\ g=g^*}} > \frac{1}{r-\mu} - q'(g^*) \tag{B.22}$$

Since the cost function q is strictly increasing, (B.22) implies that $g^* > g^u$.

□

B.3 Proof of Proposition 5 ($\kappa > 0$)

Before we prove Proposition 5, we establish an important intermediate result.

Lemma 7. *Equity holders choose to issue as much dividend as they can. That is, given g and ψ such that $\hat{\ell}(g, \psi, 0) < \bar{\ell}$, the equity holders choose*

$$m^* = \min\{\kappa, \bar{m}(g, \psi)\}, \tag{B.23}$$

where $\bar{m}(g, \psi)$ is defined in (B.7)*.*

Proof. The first-derivative of equity holders' objective function (B.12) w.r.t. m is given by

$$-p'(\hat{\ell})\ell \frac{\partial \hat{\ell}}{\partial m} > 0$$

since $p'(\hat{\ell}) < 0$ and $\partial\hat{\ell}/\partial m > 0$ (see the proof of Lemma 6). Thus, it follows that $m^* = \min\{\kappa, \bar{m}(g, \psi)\}$

□

Lemma 7 tells us that equity holders' problem can be simplified to

$$\max_{\substack{g\geq 0 \\ \psi\in[0,1]}} \frac{1+g}{r-\mu} - p(\hat{\ell})\ell - q(g) \tag{B.24}$$

$$\text{s.t. } p(\hat{\ell})\left(\hat{\ell}(1+g) - \ell\right) = \psi q(g) + m^* \tag{B.25}$$

$$p(\hat{\ell}) \geq p^B(\hat{\ell}) \tag{B.26}$$

$$m^* = \min\{\kappa, \bar{m}(g, \psi)\} \tag{B.27}$$

In other words, we can think of equity holders' problem as choosing first g and ψ and then setting m to the highest value that is feasible.

Proof of Proposition 5. The proof consists of four parts. First, we characterize the solution when $\kappa = \infty$. We refer to this solution as the "unconstrained" solution. Next, we determine $\bar{\kappa}$ such that if $\kappa \geq \bar{\kappa}$ then the unconstrained solution is attainable. Thus, for all $\kappa \geq \bar{\kappa}$ the equity payout constraint, $m \leq \kappa$, is not binding. We then show that there exists $\underline{\kappa}$ with $\underline{\kappa} < \bar{\kappa}$ such that if $\kappa \in [\underline{\kappa}, \bar{\kappa})$ then the equity holders can still attain the same payoff as in the case of $\kappa = \infty$ but with an additional restriction on their financing choices. Finally, we determine the equity holders' choices when $\kappa < \underline{\kappa}$.

When $\kappa = \infty$ then $m^* = \bar{m}(g, \psi)$ (see Lemma 7). Then the first-order derivative of equity holders' objective function w.r.t. ψ is given by

$$-p'(\hat{\ell}) \left[\frac{\partial \hat{\ell}}{\partial \psi} + \frac{\partial \hat{\ell}}{\partial m} \frac{\partial m^*}{\partial \psi} \right] \tag{B.28}$$

Since $m^* = \bar{m}(g, \psi)$, we have that

$$\frac{\partial \bar{m}(\psi, g)}{\partial \psi} = -q(g) \tag{B.29}$$

Moreover,

$$\frac{\partial \hat{\ell}}{\partial \psi} = \frac{q(g)}{p(\hat{\ell})(1+g) + p'(\hat{\ell})(\hat{\ell}(1+g) - \ell)} \tag{B.30}$$

$$\frac{\partial \hat{\ell}}{\partial m} = \frac{1}{p(\hat{\ell})(1+g) + p'(\hat{\ell})(\hat{\ell}(1+g) - \ell)} \tag{B.31}$$

Therefore, it follows that

$$p'(\hat{\ell}) \left[\frac{\partial \hat{\ell}}{\partial \psi} + \frac{\partial \hat{\ell}}{\partial m} \frac{\partial m^*}{\partial \psi} \right] = 0 \tag{B.32}$$

That is, when equity holders' equity payout choices are unconstrained they are indifferent between any $\psi \in [0, 1]$.

Fix $\psi^* \in [0, 1]$. Consider now the first-order condition that determines the optimal investment, g^*, and which is given by

$$\frac{1}{r - \mu} + p'\left(\hat{\ell}\right) \ell \left[\frac{\partial \hat{\ell}}{\partial g}\Big|_{\substack{\psi=\psi^* \\ g=g^*}} + \frac{\partial \hat{\ell}}{\partial m}\Big|_{\substack{\psi=\psi^* \\ g=g^*}} \frac{\partial m^*}{\partial g}\Big|_{\substack{\psi=\psi^* \\ g=g^*}} \right] - q'(g^*) = 0 \tag{B.33}$$

Since $m^* = \bar{m}(g, \psi)$, it follows that

$$\frac{\partial m^*}{\partial g} = \frac{1}{r - \mu} - \psi q'(g) \tag{B.34}$$

Moreover, since $m^* = \bar{m}(g,\psi)$ we know that $\hat{\ell} = \bar{\ell}$ (see Lemma 6). Therefore,

$$\frac{\partial\hat{\ell}}{\partial g}\Big|_{\substack{\psi=\psi^*\\ g=g^*}} + \frac{\partial\hat{\ell}}{\partial m}\Big|_{\substack{\psi=\psi^*\\ g=g^*}} \frac{\partial m^*}{\partial g}\Big|_{\substack{\psi=\psi^*\\ g=g^*}} = \frac{-p(\bar{\ell})\bar{\ell} + \psi^* q'(g^*) + \frac{1}{r-\mu} - \psi^* q'(g^*)}{p(\bar{\ell})(1+g^*) + p'(\bar{\ell})(\bar{\ell}(1+g^*) - \ell)} = 0, \tag{B.35}$$

where the last equality follows from the fact that $p(\bar{\ell})\bar{\ell} = 1/(r-\mu)$ (see Lemma 5). Thus, the first-order condition determining g^* simplifies to

$$\frac{1}{r-\mu} - q'(g^*) = 0 \tag{B.36}$$

implying that $g^* = g^u$. Thus, we conclude that if $\kappa = \infty$ then the solution to the equity holders' problem is given by $g^* = g^u$, $\psi^* \in [0,1]$, and $m^* = \bar{m}(g^u, \psi^*)$.

Next, we investigate for which κ the above unconstrained solution is feasible. This is the case if $\kappa \geq \bar{m}(g^u,\psi)$ for all $\psi \in [0,1]$. Note that $\bar{m}(g^u,\psi)$ is decreasing in ψ. Therefore, if we define $\bar{\kappa} \equiv \bar{m}(g^u, 0)$ then for all $\kappa \geq \bar{\kappa}$ the "unconstrained solution" is attainable.

Before proceeding further, note for all $\kappa \geq \bar{\kappa}$ equity holders' payoff is given by

$$v^*(\ell) = \frac{1+g^u}{r-\mu} - \frac{\eta}{1+\eta}\ell - q(g^u) \tag{B.37}$$

(B.37) shows us that the equity holders capture all the value of new investment and extract as much as possible from the old debt holders given the constraint that $\hat{\ell} \leq \bar{\ell}$. Moreover, note that the post-investment value of equity is independent of m, ψ, and $\hat{\ell}$.

Next, consider a situation where $\kappa < \bar{m}(g^u, 0)$ but $\kappa \geq \bar{m}(g^u, 1)$. In this case, the unconstrained solution characterized above is not feasible for some choices of ψ. However, the equity holders can still attain the payoff defined in (B.37) by choosing $g^* = g^u$, $\psi^* \in [\underline{\psi}_\kappa, 1]$, $m^* = \bar{m}(g^*, \psi^*)$, where $\underline{\psi}_\kappa$ is the unique solution to

$$\kappa = \bar{m}(g^u, \underline{\psi}_\kappa) \tag{B.38}$$

Therefore, if we define $\underline{\kappa} \equiv \bar{m}(g^u, 1)$ then the above discussion implies that for all $\kappa \geq \underline{\kappa}$ the equity holders invest the first-best amount. Finally, note that from the definition of $\bar{m}(g^u, 1)$ we have

$$\frac{\partial\underline{\kappa}}{\partial\ell} < 0, \quad \frac{\partial\underline{\kappa}}{\partial\sigma^2} > 0, \quad \frac{\partial\underline{\kappa}}{\partial r} < 0 \tag{B.39}$$

It remains to determine the equity holders' choices when $0 < \kappa < \underline{\kappa}$ (the case of $\kappa = 0$ is covered by Proposition 4). We first argue, by contradiction, that in this case the equity holders' optimal choices $\{g^*, \psi^*, m^*\}$ are such that $m^* = \kappa < \bar{m}(g^*, \psi^*)$. To see this assume, to the contrary, that $\{g^*, \psi^*, m^*\}$ are such that $m^* = \bar{m}(g^*, \psi^*) < \kappa$. Then, from Lemma 6 we know that $\hat{\ell}(g^*, \psi^*, m^*) = \bar{\ell}$, which implies that the equity holders' payoff is given by

$$\frac{1+g^*}{r-\mu} - \frac{\eta}{1+\eta}\ell - q(g^*) \tag{B.40}$$

Note that $\frac{1+g}{r-\mu} - q(g)$ is strictly increasing in g for all $g < g^u$ and strictly decreasing in g for all

$g > g^u$ and recall that since $\kappa < \underline{\kappa}$ it must be the case that $g^* \neq g^u$. Furthermore, note that if $g^* < g^u$ then the equity holders would have incentives to increase their investment from g^* to $g^* + \varepsilon$ for small $\varepsilon > 0$. This is feasible by setting

$$m' = \frac{1 + (g^* + \varepsilon)}{r - \mu} - \frac{\eta}{1 + \eta}\ell - \psi^* q(g^* + \varepsilon) \tag{B.41}$$

as long as ε is small enough so that $m' \leq \kappa$. Hence $g < g^u$ cannot be optimal. By a similar argument, if $g^* > g^u$ then the equity holders would find it optimal and feasible to decrease their investment. Thus, we conclude that a choice of $\{g^*, \psi^*, m^*\}$ such that $m^* = \bar{m}(g^*, \psi^*) < \kappa$ is not optimal.

Next, suppose that $\{g^*, \psi^*, m^*\}$ are such that $m^* = \bar{m}(g^*, \psi^*) = \kappa$. In this case, the budget constraint implies that

$$\kappa = \frac{1 + g^*}{r - \mu} - \frac{\eta}{1 + \eta}\ell - \psi^* q(g^*) \tag{B.42}$$

while the equity holders' payoff is given by

$$\frac{1 + g^*}{r - \mu} - \frac{\eta}{1 + \eta}\ell - q(g^*) \tag{B.43}$$

Using (B.42) in (B.43) we see that the equity holders' payoff can be expressed as

$$\kappa - (1 - \psi^*)\, q(g^*) \tag{B.44}$$

We now argue that the equity holders can attain a strictly higher payoff than the payoff in (B.44) by choosing $\psi' = 1$, $m' = \kappa$, and an investment g' such that g' solves the budget constraint

$$p(\hat{\ell})\left(\hat{\ell}(1 + g') - \ell\right) = \kappa + q(g') \tag{B.45}$$

with $\hat{\ell}(g', \psi', \kappa) < \bar{\ell}$. If equity holders make such choices then their payoff would be given by

$$\frac{1 + g'}{r - \mu} - p(\hat{\ell})\ell - q(g') = \frac{1 + g'}{r - \mu} - p(\hat{\ell})\hat{\ell}(1 + g') + \kappa > \kappa, \tag{B.46}$$

where the first equality follows from (B.45), while the final inequality follows from the observation that $p(\hat{\ell})\hat{\ell} < \frac{1+g}{r-\mu}$ for all $\hat{\ell} < \bar{\ell}$ (see Lemmas 1 and 5). Wwe conclude that choice of $\{g^*, \psi^*, m^*\}$ such that $m^* = \bar{m}(g^*, \psi^*) = \kappa$ cannot be optimal. It follows that we must have $m^* = \kappa < \bar{m}(g^*, \psi^*)$.

We argued above that if $\kappa \in (0, \underline{\kappa})$ then $m^* = \kappa < \bar{m}(g^*, \psi^*)$. Therefore, we have

$$\left.\frac{\partial m^*}{\partial \psi}\right|_{\substack{g=g^* \\ \psi=\psi^*}} = 0 \quad \text{and} \quad \left.\frac{\partial m^*}{\partial g}\right|_{\substack{g=g^* \\ \psi=\psi^*}} = 0 \tag{B.47}$$

This implies that, when $\kappa < \underline{\kappa}$, the first-order conditions that determine equity holders' choices of g and ψ are identical to those when $\kappa = 0$. Thus, using the same argument as in the proof of Proposition 4 we conclude that $g^* > g^u$ and $\psi^* = 1$. This concludes the proof.

□

B.4 Finite-maturity Debt

In this section, we show that our results naturally extend to the case when debt has finite maturity. In particular, we establish that Propositions 4 and 5 continue to hold in this setup. To model finite maturity debt, we follow Leland (1998) and consider debt that has no stated maturity but is continuously retired at par at a constant fractional rate $\xi > 0$. That is, at each instance of time fraction ξ of existing debt matures. It follows that $1/\xi$ is the average maturity of debt and higher ξ is associated with shorter average maturity. Each unit of debt pays a constant coupon rate of 1 and has the face value F. Finally, as in Leland (1998), we assume that equity holders are committed to always rollover their debt (i.e., keep leverage fixed), except possibly at the time of investment.[31,32]

First, we show that as long as $\xi < \infty$, that is as long as debt does not mature instantaneously, the main results establish in the paper continue to hold and equity holders continue to have incentive to overinvest. This is expected since as... Second, we show that in the case $\xi \to \infty$ (i.e., in the case that debt matures instantaneously) equity holders invest the first-best amount. This is because in that case equity holders have no ability to dilute existing debt holders' coupons as the debt . However, note that this case is not only unrealistic but also that the option value to default becomes worthless and the model becomes equivalent to the model with no default.

Proposition 8. *Propositions 4 and 5 continue to hold in the setup with finite maturity debt as long as $\xi < \infty$.*

Proof. Let T be a random default time and assume there are K units of debt outstanding. First, we compute equity holders' liability (i.e., the PDV of equity holders' promises).

$$L = \left[\int_0^\infty e^{-(r+\xi)t}(1+\xi F)dt \right] K = \frac{1+\xi F}{r+\xi} K = \varrho \frac{K}{r}, \tag{B.48}$$

where

$$\varrho \equiv \frac{r(1+\xi F)}{r+\xi} \tag{B.49}$$

and $\varrho \to 1$ as $\xi \to 0$. Therefore, $K = \frac{rL}{\varrho}$. In what follows, we use L as the state variable to make analysis easily comparable to the analysis in the main paper.

The price of finite-maturity debt is given by

$$P(Z, L; \xi) = \mathbb{E} \left[\int_0^T e^{-(r+\xi)t}(1+\xi F)dt + e^{-(r+\xi)T} \frac{\varrho V^D}{rL} \right], \tag{B.50}$$

where V^D is the value of the firm at default. Suppose that $\underline{Z}$ is the value of Z at which firm default

[31]Modeling finite maturity debt in this way leads to analytically tractable problem and has been popular both in corporate finance (see, for example, Leland (1998), DeMarzo and He (2021), He and Xiong (2012)) and in sovereign debt literature (see, for example, Chatterjee and Eyigungor (2012)).

[32]The commitment to rolling over debt is a common assumption in this literature. See the discussion in Dangl and Zechner (2021).

so that $V^D = \frac{\underline{Z}}{r-\mu}$. Then,

$$P(Z,L;\xi) = \frac{1+\xi F}{r+\xi}\left[1-\left(\frac{\underline{Z}}{Z_0}\right)^{\eta}\right] + \frac{\varrho \underline{Z}}{(r-\mu)rL}\left(\frac{\underline{Z}}{Z_0}\right)^{\eta}, \tag{B.51}$$

where

$$\eta \equiv \frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{(\mu-\frac{1}{2}\sigma^2)^2 + 2(r+\xi)\sigma^2}}{\sigma^2} \tag{B.52}$$

As $\xi \to 0$ the above price converges to the price in the baseline model.

Next, we consider the total value of the firm (total enterprise value). We have

$$TEV = \mathbb{E}\left[\int_0^\infty e^{-rt} Z_t dt\right] = \frac{Z}{r-\mu} \tag{B.53}$$

Since equity holders are residual owners, it follows that the value of equity is given by

$$V(Z,L) = TEV - P(Z,L)\frac{rL}{\varrho} = \frac{Z_0}{r-\mu} - \left[1-\left(\frac{\underline{Z}}{Z_0}\right)^{\eta}\right]L - \frac{\underline{Z}}{r-\mu}\left(\frac{\underline{Z}}{Z_0}\right)^{\eta} \tag{B.54}$$

It remains to determine the optimal default threshold. We know that $V(Z,L)$ has to satisfy the smooth-pasting condition and, hence,

$$0 = V_Z(\underline{Z}, L) = \frac{1}{r-\mu} - \left[L - \frac{\underline{Z}}{r-\mu}\right]\eta\frac{1}{\underline{Z}} \tag{B.55}$$

Solving the above equation for $\underline{Z}$, we obtain

$$\underline{Z} = \frac{\eta}{1+\eta}(r-\mu)L \tag{B.56}$$

Using the expression for $\underline{Z}$ found above in the expression for the value of equity (B.54) and simplifying, we obtain

$$V(Z,L) = \frac{Z}{r-\mu} - \left(1 - \chi\frac{1}{1+\eta}\left(\frac{L}{Z}\right)^{\eta}\right)L, \tag{B.57}$$

where χ is defined in (9). Thus, as in the baseline model, we have $V(Z,L) = v(\ell)Z$, where

$$v(\ell) = \frac{1}{r-\mu} - \left(1 - \chi\frac{1}{1+\eta}\ell^{\eta}\right)\ell \tag{B.58}$$

Note that the expression for $v(\ell)$ in (B.58) is the same as in (12). Similarly, using the expression for $\underline{Z}$ we can simplify the expression for $P(Z,L;\xi)$. In particular, we have

$$P(Z,L;\xi) = \left(1 - \chi\frac{1}{1+\eta}\ell^{\eta}\right)\frac{\varrho}{r} \tag{B.59}$$

It remains to consider the budget constraint that equity holders face at the time of investment.

As in the baseline model, let $\hat{L}$ denote post-investment liabilities. As we discussed above, if equity holders issue K' units of debt to finance their investment g then the associated increase in liabilities is given by $\frac{r}{\varrho}(\hat{L} - L)$, where $\hat{L} = \frac{\varrho}{r}(K' + K)$, and the funds the equity holders obtain are equal to

$$\left(1 - \chi \frac{1}{1+\eta} \ell^{\eta}\right)(\hat{L} - L) \tag{B.60}$$

Therefore, the equity holders' budget constraint (divided by Z) is given by

$$\left(1 - \chi \frac{1}{1+\eta} \ell^{\eta}\right)(\hat{\ell}(1+g) - \ell) = \psi q(g) + m, \tag{B.61}$$

We see that compared to the baseline model, the only difference is that η now depends on ξ. Thus, Propositions 4 and 5 remain valid in the model with finite maturity debt. □

Proposition 9. *If $\xi \to \infty$ then the equity holders invest the first-best amount.*

Proof. Note that Equation (B.56) implies that equity holders continue to operate the firm if and only if $\ell < \frac{1+\eta}{\eta(r-\mu)}$. It follows that for all $\eta > 1$ we have

$$\chi \ell^{\eta} < 1 \tag{B.62}$$

Since $\xi \to \infty$ then $\eta \to 1$ and $\chi \to 0$ (the latter as long as $r - \mu$<1 which is the case for realistic values), we have that $\lim_{\xi \to \infty} \chi \ell^{\eta} = 0$. It follows that the equity holders' problem becomes

$$\max_{g,\psi} \frac{1+g}{r-\mu} - \ell - q(g) \qquad \text{s.t.} \quad \hat{\ell}(1+g) - \ell = \psi q(g) + m \tag{B.63}$$

It follows that the equity holders always invest the first-best amount and are indifferent between financing investment and equity payouts with any mix of debt and equity. □

B.5 Cash holdings

Suppose the firm is allowed to hold cash on its balance sheet. One may wonder whether the firm can use an increase in its cash holding as a way to dilute existing debt holders instead of resorting to inefficient investment. In this section, we show that this is not the case and that equity holders would not have incentives to raise cash in our baseline model.

To see this, consider our baseline model with no bankruptcy costs and $\kappa = 0$ (i.e, no equity payouts) but assume that equity holders at time 0 can raise cash.[33] Furthermore assume that cash (1) has to be left on firm's balance sheet (i.e., equity holders cannot withdraw it/use it) and (2) is distributed among all debt holders on pari passu basis at the time of default. We also assume that

[33] We are isolating the choice of cash from the other investments because they are easily separable, but it could be a joint decision. We choose the simple $\kappa = 0$ case because equity holders always want to maximize direct equity payouts and the presence of cash does not change this incentive.

cash earns zero rate of return.[34] Note that under these assumptions equity holders do not benefit directly from raising cash, only indirectly via increase in leverage. It follows that conditional on leverage, the value of equity and equity holders' optimal default decisions are unchanged. Initially the firm is assumed to hold no cash.

Let $P(L, Z, C)$ denote the price of unit of debt issued by firm with liability L, cash flows, Z, and cash holdings, C. Then,

$$P(L, Z, C) = \frac{1}{r}\left[\left(1 - \chi\frac{1}{1+\eta}\ell^{\eta}\right) + \frac{C}{L}\chi\ell^{\eta}\right] = \frac{p(\ell, c)}{r}, \tag{B.64}$$

where $\frac{C}{L}\chi\ell^{\eta}$ captures the present value of cash that a debt holder of a unit of debt will receive at the time of default and $c = C/Z$. It follows that the value of equity of a firm with cash flows Z, liability L, and cash holding C can be written as

$$V(\ell, Z) = \left[\frac{1}{r-\mu} - p(\ell, 0)\ell\right] Z \tag{B.65}$$

That $V(\ell, Z)$ is independent of C follows from the observation that only care about the value of cash flows that debt holders were promised to obtain.

Now suppose that at time 0 equity holders decide how much cash to raise. Let $\hat{c}$ and $\hat{\ell}$ be firm's cash holdings and leverage after the firm raised cash. Then, just after the equity holders made their decisions, the value of equity is given by

$$V(\hat{\ell}, Z) = \left[\frac{1}{r-\mu} - p(\hat{\ell}, 0)\hat{\ell}\right] Z \tag{B.66}$$

The equity holder's budget constraint is given by

$$p(\hat{\ell}, \hat{c})(\hat{\ell} - \ell) = \hat{c} \tag{B.67}$$

Note that (B.67) implies that $\partial\hat{\ell}/\partial\hat{c} > 0$. Now, differentiating the value of equity in (B.66) w.r.t. $\hat{c}$ we obtain

$$\frac{\partial V(\hat{\ell}, Z)}{\partial \hat{c}} = \frac{\partial \hat{\ell}}{\partial \hat{c}}\left(\frac{p(\hat{\ell}, 0)}{\partial \hat{\ell}}\hat{\ell} + p(\hat{\ell}, 0)\right) = -\frac{\partial \hat{\ell}}{\partial \hat{c}}[1 - \chi\hat{\ell}^{\eta}] < 0, \tag{B.68}$$

where the last inequality follows the fact that $1 - \chi\hat{\ell}^{\eta} > 0$ (see Corollary 3). This establishes that the value of equity is strictly decreasing in $\hat{c}$ and so equity holders have no incentives to raise cash. We summarize the above result in the following lemma.

Lemma 8. *The equity holders never choose to raise cash.*

Why do equity holders have incentive to engage in inefficient investment but do not have incentive

[34] Allowing cash earn low but positive rate of return does not change the result. In addition, using the collateralized debt interpretation rather than parri passu would be identical since the existing collateral claims are not diluted in bankruptcy with our mechanism.

to raise cash? To understand this note that when equity holders raise cash they do decrease the value of existing coupon claims. In particular, in contrast to inefficient investment, equity holders only benefit indirectly from raising cash.

Moreover, raising cash also benefits existing debt holders by moving forward the timing of default and even increasing the value of their default claims by providing additional cash in default.[35] This limits the dilution mechanism. It also implies that new debt holders will recover part of the funds they provided to equity holders. Therefore, new debt holders require high compensation for their funds. As such the cost of raising cash exceeds the benefit due to dilution of existing debt holders.

Instead, when equity holders engage in inefficient investment the equity holders also benefit from it directly via higher cash flows the firm earns till the time of default. This direct benefit together with indirect benefit due to dilution of existing debt holders makes equity holders overinvest. If investment was costly but unproductive (i.e., had no effect on cash flows) then equity holders would never make such investment for similar reasons why they do not raise cash (i.e., indirect benefit via dilution being too small to compensate for the cost of raising funds via debt).

B.6 Bankruptcy Costs

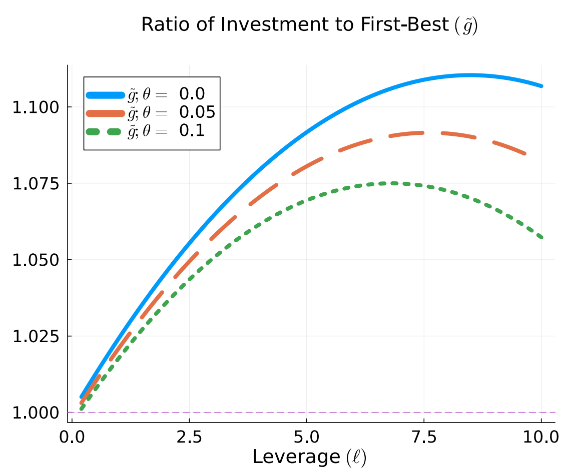

Figure 5: **Investment relative to first-best ($\tilde{g}$) against preexisting leverage (ℓ) for the baseline model with a single financing-investment opportunity.** Each line corresponds to a different level of deadweight bankruptcy costs (θ). The case $\theta = 0$ corresponds to no bankruptcy costs. Model parameters are discussed in Appendix F.

Figure 5 shows that for empirically plausible bankruptcy costs that the incentive to double-sell some of the existing debt holders' coupon claims dominates and equity holders overinvest for all levels of leverage.

[35]If default claims were entirely collateralized and old default claims could not benefit from the cash in the firm—as in our collateralized interpretation of the model, the results would be the same. The key is that the equity holders cannot gain from rearranging cashflows between new and old claimants.

B.7 Model with Two Investment Opportunities

In this section, we provide the proof of Proposition 6. Note that behavior of all agents from time T onward is the same as in our benchmark model with single investment opportunity. Therefore, from time 0 perspective, the expected payoff to equity holders onward is

$$e^{-rT} Z_T \left[\frac{1+g_T^*}{r-\mu} - p_T(\hat{\ell}_T)\hat{\ell_T} \right] \tag{B.69}$$

where $p_T(\cdot)$ is the price at time T of a unit of debt issued by equity holders, g_T^* is the optimal investment choice of equity holders' at time T determined by Equation (26), and $\hat{\ell}_T$ is the post-investment leverage determined by the budget constraint (23).

We begin by computing the price of a unit of debt issued at time 0.

Lemma 9. *Let $\hat{\ell}_0$ be the post-investment leverage at time 0. The price of a bond that is issued at time 0 to finance the initial investment is*

$$P_0(\hat{\ell}_0) = \frac{p(\hat{\ell}_0)}{r} = \frac{1}{r}\left(1 - e^{-rT}\frac{\chi}{1+\eta}(\hat{\ell}_T(\hat{\ell}_0))^\eta\right), \tag{B.70}$$

where $\hat{\ell}_T(\hat{\ell}_0)$ is time T post-investment leverage as a function of $\hat{\ell}_0$.

Proof. Recall that each unit of debt issued at time 0 pays one unit for all $t \in [0, T]$ (since there is no default till T). After that, debt pays one unit till default and upon default a holder of one unit of debt has a claim to firm's value in default. Therefore, each unit of debt is associated with expected payoff

$$\int_0^T e^{-rt}dt + e^{-rT}\mathbb{E}\left[\int_T^\tau e^{-r(t-T)}dt + e^{-r(\tau-T)}\frac{V(\underline{Z}(\hat{L}_T))}{r\hat{L_T}}\right],$$

where τ is the random default time, $V(\underline{Z}(\hat{L}_T))$ is the default value of the firm, and $r\hat{L}_T$ is the amount of outstanding debt after investment at time T. Therefore, the price of one unit of debt issued at time 0 is given by

$$P_0(\hat{\ell}_T) = \frac{1}{r}\left[1 - e^{-rT}\right] + e^{-rT}\left[1 - \frac{\chi}{1+\eta}\hat{\ell}_T^\eta\right] = 1 - e^{-rT}\frac{\chi}{1+\eta}\hat{\ell}_T^\eta,$$

where we used the characterization of equity holders' default decision in the model with single investment opportunity.

Finally, since there is no uncertainty in the model till time T, it follows that from time 0 perspective, once $\hat{\ell}_0$ is determined, the investment at time T and post-investment leverage $\hat{\ell}_T$ are also determined. Therefore, $\hat{\ell}_T$ is a function of $\hat{\ell}_0$, which justify using $\hat{\ell}_0$ as the relevant state variable for the price of debt at time 0.

□

With the above result in hand, we now provide the proof of Proposition 6.

Proof of Proposition 6. As shown above (see Equation (B.69)) that from time 0 perspective, the expected payoff to equity from time T onward is given by

$$e^{-rT}Z_T\left[\frac{1+g_T^*}{r-\mu}-p_T(\hat{\ell}_T)\hat{\ell_T}(1+g_T^*)\right]$$

To payoff to equity holders between time 0 and T is given by

$$\begin{aligned}\int_0^T e^{-rt}(Z_t-r\hat{L}_0)dt &= \int_0^T e^{-rt}\left((1+g)Z_0e^{\mu t}-\hat{L}_0\right)dt\\ &= (1+g_0)\frac{Z_0}{r-\mu}\left(1-e^{-(r-\mu)T}\right)-\hat{L}_0\left(1-e^{-rT}\right)\end{aligned}$$

where $\hat{L}_0$ are post-investment firm's liability and rL_0 is the quantity of outstanding debt.

Putting these two observations together, we see that the expected payoff to equity holders at time 0, conditional on investment g_0, is given by

$$(1+g_0)\frac{Z_0}{r-\mu}\left(1-e^{-(r-\mu)T}\right)-\hat{L}_0\left(1-e^{-rT}\right)+e^{-rT}Z_T\left[\frac{1+g_T^*}{r-\mu}-p_T(\hat{\ell}_T)\hat{\ell_T}(1+g_T^*)\right]$$

Recognizing that $Z_T=e^{\mu T}(1+g_0)Z_0$ and factoring out $(1+g_0)Z_0$, we obtain

$$(1+g_0)Z_0\left\{\frac{1}{r-\mu}\left(1-e^{-(r-\mu)T}\right)-\hat{\ell}_0\left(1-e^{-rT}\right)+e^{-(r-\mu)T}\left[\frac{1+g_T^*}{r-\mu}-p_T(\hat{\ell}_T)\hat{\ell_T}(1+g_T^*)\right]\right\} \tag{B.71}$$

From Lemma 9 the price of unit of debt issued at time 0 normalized by r is given by $p_0(\hat{\ell}_0)=1-e^{-rT}\frac{\chi}{1+\eta}[\hat{\ell}_T(\hat{\ell}_0)]^\eta$. Therefore, we note that

$$\begin{aligned}\hat{\ell}_0\left(1-e^{-rT}\right)=&\hat{\ell}_0\left(1-e^{-rT}\frac{\chi}{1+\eta}\hat{\ell}_T^\eta+e^{-rT}\frac{\chi}{1+\eta}\hat{\ell}_T^\eta-e^{-rT}\right)\\ =&\hat{\ell}_0\left[1-e^{-rT}\frac{\chi}{1+\eta}\hat{\ell}_T^\eta\right]-e^{-rT}\hat{\ell}_0\left[1-\frac{\chi}{1+\eta}\hat{\ell}_T^\eta\right]\\ =&p_0(\hat{\ell}_0)\hat{\ell}_0-p_T(\hat{\ell}_T)(\ell_T)\hat{\ell}_0\end{aligned}$$

Substituting this into Equation (B.71) and rearranging, we arrive at

$$(1+g_0)Z_0\left\{\frac{1}{r-\mu}\left(1-e^{-(r-\mu)T}\right)-p_0(\hat{\ell}_0)\hat{\ell}_0+e^{-(r-\mu)T}\left[\frac{1+g_T^*}{r-\mu}-p_T(\hat{\ell}_T)(\hat{\ell_T}(1+g_T^*)-\hat{\ell}_0)\right]\right\}$$

Finally, given equity holders' investments g_T^* at time T and g_0 at time 0 the budget constraint in period T is given by

$$p_T(\hat{\ell}_T)(\hat{\ell}_T(1+g_T^*)-\ell_T)=q(g_T^*)$$

while the budget constraint in period 0 is given by

$$p_0(\hat{\ell}_0)(\hat{\ell}_0(1+g_0)-\ell_0)=q(g_0)$$

Substituting these budget constraints into the expression for the value of equity in Equation (C.1), we obtain the desired expression.

□

Appendix C Repeated Investment Derivations

This section derives the ODEs for a repeated investment decisions, which introduces a controlled jump-process. Assume that upon an arrival of an financing-investment opportunity, the state jumps to a deterministic function of the current state, $\hat{\ell}(\ell)$. Define the jump size as $\tilde{g}(\ell) \equiv \hat{\ell}(\ell) - \ell$. Then the SDE for ℓ is

$$d\ell_t = (\sigma^2-\mu)\ell_t \mathrm{d}t + \sigma\ell_t d\mathbb{W}_t + \left(\hat{\ell}(\ell)-\ell\right) d\mathbb{N}_t \tag{C.1}$$

where $\mathbb{N}_t$ is a homogeneous Poisson process with arrival rate $\lambda \geq 0$.

Firm's HJBE First, we will derive the HJBE in ℓ-space without the jumps, and add them. Set $V(Z,L) = Zv(L/Z)$ and differentiate w.r.t. Z

$$\boldsymbol{\partial}_Z V(Z,L) = v(L/Z) - \frac{L}{Z}\boldsymbol{\partial}_\ell v(L/Z) = v(\ell) - \ell\boldsymbol{\partial}_\ell v(\ell) \tag{C.2}$$

$$\boldsymbol{\partial}_{ZZ} V(Z,L) = \frac{L^2}{Z^3}\boldsymbol{\partial}_{\ell\ell} v(L/Z) = \frac{1}{Z}\ell^2\boldsymbol{\partial}_{\ell\ell} v(\ell) \tag{C.3}$$

Use the ODE in (6), divide by Z, and use the above derivatives to obtain

$$(r-\mu)v(\ell) = 1 - r\ell - \mu\ell\boldsymbol{\partial}_\ell v(\ell) + \frac{\sigma^2}{2}\ell^2\boldsymbol{\partial}_{\ell\ell} v(\ell) \tag{C.4}$$

Default Decision The notation denotes $\cdot|_\ell$ as the evaluation of a function at ℓ. We can write the DVI for their stopping problem as

$$u(\ell) \equiv 1 - r\ell \tag{C.5}$$

$$\mathcal{L} \equiv r - \mu + \mu\ell\boldsymbol{\partial}_\ell - \frac{\sigma^2}{2}\ell^2\boldsymbol{\partial}_{\ell\ell} - \lambda\left(\cdot|_{\ell+\tilde{g}(\ell)} - \cdot|_\ell\right) \tag{C.6}$$

$$0 = \min\{\mathcal{L}v(\ell) - u(\ell), v(\ell)\} \tag{C.7}$$

$$\lim_{\ell\to 0}\boldsymbol{\partial}_\ell v(\ell) = 0 \tag{C.8}$$

$$\lim_{\ell\to\infty}\boldsymbol{\partial}_\ell v(\ell) = 0 \tag{C.9}$$

We would numerically find a $\bar{\ell}$ which fulfills the indifference point, and then find the value of liquidation per unit of PV of liabilities is

$$v^{\text{liq}} \equiv (1-\theta)\frac{\lim_{\ell\to 0} v(\ell)}{\bar{\ell}} \tag{C.10}$$

Bond Pricing The price of a bond, $P(Z,L)$ pays 1 unit until default. The ODE in the continuation region without jumps is

$$rP(Z,L) = 1 + \mu Z \boldsymbol{\partial}_Z P(Z,L) + \frac{\sigma^2}{2} Z^2 \boldsymbol{\partial}_{ZZ} P(Z,L) \tag{C.11}$$

Take the defintion $rP(Z,L) \equiv p(L/Z)$ and differentiate with respect to Z

$$r\boldsymbol{\partial}_Z P(Z,L) = -\frac{1}{Z}\ell \boldsymbol{\partial}_\ell p(\ell) \tag{C.12}$$

$$r\boldsymbol{\partial}_{ZZ} P(Z,L) = \frac{1}{Z^2}\left(2\ell\boldsymbol{\partial}_\ell p(\ell) + \ell^2 \boldsymbol{\partial}_{\ell\ell} p(\ell)\right) \tag{C.13}$$

Multiply by r and substitute the derivatives into (C.11)

$$rp(\ell) = r + (\sigma^2 - \mu)\ell\boldsymbol{\partial}_\ell p(\ell) + \frac{\sigma^2}{2}\ell^2\boldsymbol{\partial}_{\ell\ell} p(\ell) \tag{C.14}$$

In default, the bond is entitled to a share of rL units of the liquidation value $V((1-\theta)\underline{Z}(L), 0)$, hence $P(\bar{\ell}) = \frac{V((1-\theta)\underline{Z}(L),0)}{rL}$. Divide by r and use the definitions of v^{liq} and $p(\cdot) = P(\cdot)/r$ to find that the boundary condition is $p(\bar{\ell}) = v^{\text{liq}}$.

Summarizing, bond pricers take v^{liq} and $\bar{\ell}$ as given, and then solve

$$\mathcal{L} \equiv r - (\sigma^2-\mu)\ell\boldsymbol{\partial}_\ell - \frac{\sigma^2}{2}\ell^2\boldsymbol{\partial}_{\ell\ell} - \lambda\left(\cdot|_{\ell+\tilde{g}(\ell)} - \cdot|_\ell\right) \tag{C.15}$$

$$\mathcal{L}p(\ell) = r \tag{C.16}$$

$$\lim_{\ell\to 0}\boldsymbol{\partial}_\ell p(\ell) = 0 \tag{C.17}$$

$$p(\bar{\ell}) = v^{\text{liq}} \tag{C.18}$$

where the lower boundary is equivalent to a reflecting barrier as $Z\to\infty$ for a given L, and the upper boundary is the liquidation absorbing barrier.

Investment Finally, the objective function of the firm at every arrival point λ remains to maximize the equity value. Given an equilibrium $p(\ell)$ and $v(\ell)$ functions—consistent with the optimal jump process, the agent solves the problem described by (30).

First-Best The first-best is derived through a guess-and-verify approach. First, guess that the user would choose a constant g due to the homotheticity of the problem. With that, the unnormalized Bellman equation (with jumps) is

$$rV(Z) = Z + \mu Z V'(Z) + \frac{\sigma^2}{2}Z^2 V''(Z) + \lambda \max_g \left\{V((1+g)Z) - V(Z) - \zeta\frac{g^2}{2}Z\right\} \tag{C.19}$$

Take the first-order condition

$$\zeta g Z = Z V'((1+g)Z) \tag{C.20}$$

Guess the solution to the problem is $V(Z) = AZ$ for an undetermined Z, and subtitute into the (C.19) and solve for A to find,

$$A = \frac{1 - \frac{1}{2}\zeta g^2 \lambda}{-g\lambda - \mu + r} \tag{C.21}$$

Similarly, substitute the guess into (C.20) to find $g = \frac{A}{\zeta}$. Use this expression to eliminate A in (C.21), solve the quadratic for g, and choose the positive root to find,

$$g^u = \frac{1}{\zeta(r-\mu)\left(\frac{1}{2}\left(\sqrt{1 - \frac{2\lambda}{\zeta(r-\mu)^2}} - 1\right) + 1\right)} = \frac{2}{\sqrt{\zeta\left(\zeta(r-\mu)^2 - 2\lambda\right)} + \zeta(r-\mu)} \tag{C.22}$$

In addition, given that $V(Z) = AZ$ and noting that $V(Z,0)$ we obtain $v(0) = \frac{V(Z,0)}{Z} = A$. Consequently, for a default threshold $\bar{\ell}$, the liquidation value per unit of defaultable console in (C.10) is,

$$v^{\text{liq}} = \frac{1-\theta}{\bar{\ell}} \frac{1 - \frac{1}{2}\zeta (g^u)^2 \lambda}{r - \mu - g^u \lambda} \tag{C.23}$$

When $\lambda = 0$, these all nest the $g^u = \frac{1}{\zeta(r-\mu)}$ case.

Appendix D Numerical Methods

In this section we describe the numerical methods used to solve for the equilibrium with repeated investment opportunities using upwind finite difference methods. In particular, we need to find a $p(\cdot)$ and $v(\cdot)$ consistent with the firm's optimal investment choice (i.e., $\hat{\ell}(\cdot)$ and $g(\cdot)$).

Bounds on the State Space The change of variables into ℓ space described in Appendix C creates a discontinuity in the solution at $\ell = 0$ (though not in the original Z, L space) so we will restrict our solutions to the $\ell > 0$ case and choose a lower bound $\ell_{\min} > 0$. This number is chosen to be sufficiently small that it will not distort the solution of the ODE. In addition, to discretize we need to truncate the state space at some $\ell_{\max}$. The exact value will not be important as long as it is above the equilibrium $\bar{\ell}$ threshold during any iteration of the algorithm.

Beyond setting the range of the approximated value functions, these bounds are used as an artificial reflecting barrier at $\ell_{\min}$ and $\ell_{\max}$ as a truncated version of the asymptotic boundary conditions (i.e., $v'(\ell_{\min}) = 0$ and $v'(\ell_{\max}) = 0$ for the firm's default problem and $p'(\ell_{\min}) = 0$ for the bond pricing).[36]

[36] These artificial reflecting barriers at distant ends of the state space are the standard approach to solving discretized SDEs on an unbounded space.

Discretizing the State Space Define a grid $\vec{\ell}_{\text{ex}} \in \mathbb{R}^{M+2}$ with elements from $\ell \in [\ell_{\min}, \ell_{\max}]$. In our baseline, we use $\ell_{\min} = 0.01$ and $\ell_{\max} = 50.0$ with $M = 500$ grid points. The step-size of the discretization as $\Delta \equiv \vec{\ell}_2 - \vec{\ell}_1$, is assumed constant.

The first and last elements of this discretization of the state-space are used as "ghost nodes" to solve the differential equations when carefully applying boundary conditions. Consequently, in some cases we will solve using the interior of the set, $\vec{\ell} \equiv \vec{\ell}_{\text{ex}}[2 : M+1]$.

The second subset of this domain is that the bond pricing differential equation has an absorbing barrier at $\bar{\ell}$. In that case, given if the index of the element within $\vec{\ell}$ is $\iota_{\bar{\ell}}$ then define the subset of the space as $\vec{\ell}_{\bar{\ell}} \equiv \vec{\ell}_{\text{ex}}[1 : \iota_{\bar{\ell}}] \in \mathbb{R}^{M_{\bar{\ell}}+2}$. Note that in this formulation we will keep the "ghost nodes" in order to apply boundary conditions directly in the linear system.

When solving the $v(\cdot)$ differential equation and form a linear complementarity problem (LCP), we need to be able to discretize the the infinitesimal generators on the interior of the domain, directly imposing the boundary conditions. In the case of that ODE, the boundary conditions are $v'(\ell_{\min}) = 0$ and $v'(\ell_{\max}) = 0$—which imposes a reflecting barrier at the boundaries. Without an equation along these lines, the problem is ill-posed. In this discretization at the bottom of the operator, the "ghost node" at $\ell_{\min}$ occurs in the first element of the grid, in which case the discretization of the boundary condition becomes $\Delta(v_2 - v_1) = 0$, with a similar condition at the top of the grid. Following standard upwind finite-difference methods, due to the linearity of the differential operators in the DVI, the boundary conditions can be applied operator-by-operator.

Fixed Point The algorithm operates by finding a fixed point on the $\hat{\ell}(\ell)$ policy which arises from the firm's optimal investment choice at the time of an investment opportunity.

For clarity, denote $\hat{\ell}$ as a vector of the same length as the grid $\vec{\ell}$. Similarly, define $\vec{v}$ and $\vec{p}$ as the value functions and price functions approximated on the grid $\vec{\ell}$. Define the fixed point operator $T : \hat{\ell} \to \hat{\ell}$ by:

1. Taking the $\hat{\ell}$ as given, solve the firm's continuation $\vec{v}$ with associated default threshold $\bar{\ell}$ fulfilling the DVI in (C.6) to (C.9).
2. Taking the $\hat{\ell}$ and $\bar{\ell}$ as given, solve for the $\vec{p}$ fulfilling the bond-pricing equation (C.15) to (C.18).
3. For each $\ell \in \vec{\ell}$, and taking the $\vec{v}$ and $\vec{p}$ from the previous steps as given, solve the for the optimal investment decision choice from (30). Return the resulting optimal $\hat{\ell}(\ell)$ evaluated for $\ell \in \vec{\ell}$ choice.

We then use a fixed point algorithm, such as Anderson Acceleration, to find a point $\hat{\ell}$ such that $T(\hat{\ell}) = \hat{\ell}$. Given this equilibrium jump process the resulting $v(\cdot), g(\cdot), p(\cdot)$ can then be calculated as in the above sub-steps of the iteration.

We solve both the equity holder's optimal default choice and the bond pricing using upwind finite difference methods. To do this, we need to discretize the infinitesimal generators of the jump diffusion process. As these are linear differential operators, we can discretize them separately and combine in the solution to the appropriate ODEs.

Discretized Infinitesimal Generators of the Diffusion Process The discretization of the diffusion terms are relatively straightforward because they do not depend on equilibrium objects.

The only technical difficulty is in correctly applying boundary conditions, which will be different for the case of $p(\cdot)$ and $v(\cdot)$.[37]

For the drift process, to discretize the $\boldsymbol{\partial}_\ell$ term, it is important to use "upwind" finite differences. i.e., backward first-differences if the drift is positive, and forward finite-differences the drift is negative. Here we assume that, as in our baseline parametrization, when discretizing the operator for the value function we have $\mu < 0$—which leads to forward differences, and when discretizing the operator for the $v(\cdot)$ solution $\sigma^2 - \mu < 0$ which leads to backward-differences in the case of $p(\cdot)$.

$$L_1^v \equiv \frac{1}{\Delta}\begin{bmatrix} -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{M\times M} \tag{D.1}$$

$$L_1^p \equiv \frac{1}{\Delta}\begin{bmatrix} -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \end{bmatrix} \in \mathbb{R}^{M_{\bar\ell}\times M_{\bar\ell}+2} \tag{D.2}$$

As discussed above, the definition of L_1^v imposes the boundary conditions $v'(\ell_{\min}) = 0$ and $v'(\ell_{\max}) = 0$ whereas the L_1^p will manually apply the boundary conditions (by adding in 2 additional equations and solving a system of $M_{\bar\ell} + 2$ equations in $M_{\bar\ell} + 2$ unknowns).

The discretization of the second-order operator can use central differences (as upwind only applies to the advective term). In that case, we can discretize the operators—imposing the boundary

[37]For some papers, e.g. Huang (1998), they seem to sidestep this issue and simply truncate the differential operators without directly imposing any boundary conditions. This likely does something similar to applying a reflecting barrier on the diffusion process. Here we will be more careful to apply boundary conditions which is necessary to correctly solve the bond pricing function given the default policy, but the results are numerically indistinguishable when solving for the value function with a large enough grid.

conditions on the $v(\cdot)$ operator directly as

$$L_2^v \equiv \frac{1}{\Delta^2}\begin{bmatrix} -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & -2 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix} \in \mathbb{R}^{M\times M} \tag{D.3}$$

$$L_2^p \equiv \frac{1}{\Delta^2}\begin{bmatrix} 1 & -2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & -2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \end{bmatrix} \in \mathbb{R}^{M_{\bar{\ell}}\times M_{\bar{\ell}}+2} \tag{D.4}$$

Discretized Infinitesimal Generators of the Jump Process Unlike the diffusion process, the jump process—$\mathcal{L}_{\text{jump}} \equiv \left(\cdot|_{\ell+\tilde{g}(\ell)} - \cdot|_{\ell}\right)$—depends on the equilibrium $\hat{\ell}(\ell)$ policy (which we are using for the fixed point algorithm). To implement this, we find the closest grid point within $\vec{\ell}$ to the $\hat{\ell}$ value. If ℓ_i is the ith element of $\vec{\ell}$ then denote $j(i)$ is the index (within $\vec{\ell}$) of the closest value of $\hat{\ell}_i$.

With this, we can define the discretized jump operators by,

$$L_{\text{jump}}[i,j] \equiv \frac{1}{\Delta} \times \begin{cases} -1 & \text{if } i = i \\ 1 & \text{if } i = j(i) \\ 0 & \text{otherwise} \end{cases} \tag{D.5}$$

where $L^v_{\text{jump}} \in \mathbb{R}^{M\times M}$ and $L^p_{\text{jump}} \in \mathbb{R}^{M_{\bar{\ell}}\times M_{\bar{\ell}}+2}$. The difference between these is purely the offset of $j(i)$ for where the jumps occur.[38]

As before, the boundary conditions are applied in the case of L_{jump}, and while L^p_{jump} could jump to ghost nodes, in equilibrium this will not occur as the $\hat{\ell}$ policy always jumps weakly forwards, and never jumps into the default range.

Value Function and Default Policy See Huang (1998) and `https://benjaminmoll.com/codes/` for a detailed discussion of numerical methods for solving an optimal stopping problem using differential variational inequalities. Given the discretizations of the operators above, we can directly translate (C.6) and (C.7) into matrices and vectors. The boundary conditions, (C.8) and (C.9) were imposed automatically by the discretization process.[39]

[38]In the case of the L^p_{jump} operator it needs to take into account jumps to and from the ghost nodes—which will never happen in equilibrium.

[39]This formulation is equivalent to stacking up 2 extra equations for the boundaries and performing two steps of Gaussian elimination. Without this step, the problem would instead by a linearly-constrained LCP.

$$\vec{u} \equiv \mathbf{1} - r\vec{\ell} \tag{D.6}$$

$$L^v \equiv (r-\mu)\mathbf{I} + \mu \,\mathrm{diag}(\vec{\ell}) \cdot L_1^v - \frac{\sigma^2}{2}\mathrm{diag}(\vec{\ell}) \cdot \mathrm{diag}(\vec{\ell}) \cdot L_2^v - \lambda L_{\mathrm{jump}}^v \tag{D.7}$$

$$0 = \min\{L^v \cdot \vec{v} - \vec{u}, \vec{v}\} \tag{D.8}$$

Where $\mathbf{I}$ is the identity matrix, $\mathbf{1}$ is a vector of ones, and $\mathrm{diag}(\cdot)$ creates a diagonal matrix from a vector. The final equation is a linear complementarity problem (LCP) which can be solved for $\vec{v} \in \mathbb{R}^M$.[40]

Given the solution, we can find the index of the default threshold $\bar{\ell}$ by finding the smallest index where the DVI is binding, i.e. $\iota_{\bar{\ell}} \equiv \min_{i=1,\dots M}\{\vec{v}_i < 0\}$.

Bond Pricing Unlike the problem of finding the value function, bond prices have no ex-post decisions and are simply paid coupons until default occurs when $\ell = \bar{\ell}$. Given a $\bar{\ell}$ default policy, the resulting v^{liq} liquidation value of the firm, and the discretizations of operators above, we can directly translate (C.15) to (C.18) into matrices and vectors. Unlike in the previous section, we need to apply the boundary conditions directly and solve with the ghost node at $\bar{\ell}$ which acts as an absorbing barrier and with the artificial reflecting barrier at $\ell_{\min}$. Using the discretized operators, the solution for the bond prices $\vec{p} \in \mathbb{R}^{M_{\bar{\ell}}+2}$ is a linear system

$$L^p \equiv r\mathbf{I}^p - (\sigma^2 - \mu)\mathrm{diag}(\vec{\ell}) \cdot L_1^p - \frac{\sigma^2}{2}\mathrm{diag}(\vec{\ell}) \cdot \mathrm{diag}(\vec{\ell}) \cdot L_2^p - \lambda L_{\mathrm{jump}}^p \tag{D.9}$$

$$B^p \equiv \begin{bmatrix} -1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \quad A \equiv \begin{bmatrix} L^p \\ B^p \end{bmatrix} \tag{D.10}$$

$$\vec{b} \equiv \begin{bmatrix} r & r & \dots & r & 0 & v^{\mathrm{liq}} \end{bmatrix}^\top \in \mathbb{R}^{M_{\bar{\ell}}+2} \tag{D.11}$$

$$A \cdot \vec{p} = \vec{b} \tag{D.12}$$

Where $\mathbf{I}^p \equiv \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}$, i.e. an identity matrix with two columns of stacked zeros on each side. Note that B^p is imposing the discretized boundary conditions $\Delta(p_1 - p_2) = 0$ and $p_{\iota_{\bar{\ell}}} = v^{\mathrm{liq}}$ through (D.10) and (D.12).

Optimal Investment Decision Given the $\vec{v}$ and $\vec{p}$ functions, we use cubic spline interpolation to give approximate $v(\cdot)$ and $p(\cdot)$ functions which can be evaluated outside of grid points, and with $p(\ell) = 0$ for $\ell \geq \bar{\ell}$ since it is in the default region.

Then the optimal investment decision is simply a nonlinear optimization problem with objective given by (30) and constraints given by (23) and (24). This optimization problem is solved for each $\ell \in \vec{\ell}_{\mathrm{ex}}$ and the vector of resulting $\hat{\ell}(\ell)$ is returned for the for the fixed-point calculation of T.

[40] We use the the PATH solver (Dirkse and Ferris, 1995), but other algorithms such as Lemke's algorithm could be used.

Appendix E Covenants

There is a large existing literature that documents that covenants are commonly used to protect existing debtholders (see Smith and Warner (1979), Billett et al. (2007), Chava et al. (2010), Reisel (2014)).[41] Below, we discuss how commonly used covenants would affect our results. Our main conclusion is that, depending on their type, covenants are unlikely to resolve the issues and even further support our modeling choices.

Restrictions on payouts: We think of restrictions on payouts being captured by our model parameter κ. From Figure 5 we see that as these restrictions become tighter, they tend to mitigate underinvestment for low leverage firms but tend to exacerbate overinvestment for the remaining firms. While it is theoretically possible to devise payout restrictions that would restore first-best investment, such covenants would have to be state-contingent. It is therefore unlikely that optimality could be restored through such covenants in practice.

Secured debt restrictions: Secured debt restrictions (often referred to as negative pledge covenants) prohibit firms from issuing secured debt, unless all pre-existing debt also obtains a proportional claim to the same collateral. This covenant is typically used by unsecured lenders to protect themselves from dilution, though it may be difficult to enforce in practice (see Donaldson et al. (2019)). In our benchmark model, all debt already has equal priority claims in bankruptcy, thus this type of covenant does not have any bite.

Restrictions on leverage: Restrictions on leverage are relatively common covenants particularly for non-investment grade firms (see Billett et al. (2007)). However, in our model investment distortions occur even at low to medium leverage levels (firms that are unlikely to violate leverage restrictions). Moreover, the highest leverage firms in our model tend to even decrease leverage while overinvesting. It is therefore unlikely that covenants that restrict leverage can correct equity holders' investment incentives in our model.

Senior debt restrictions: Senior debt restrictions prohibit the firms from issuing senior debt. These types of covenants are empirically extremely rare affecting only 0.2% of firms in Billett et al. (2007) post-2000 sample. In our model, this would imply that all new debt would have to be junior compared to the existing debt. As we show in the Appendix, this covenant would resolve the issue of overinvestment but at the cost of firms underinvesting for all levels of leverage.

Appendix F Calibration

To discipline parameters for the exposition, we calibrate to moments from the firm dynamics (Sterk et al. (2021)) and investment spikes (Gourio and Kashyap (2007)) literature.

To guide the relevant range of ℓ, Palomino et al. (2019) report an average interest coverage ratio of around 4 for the period 1970-2017. They also find that an interest coverage ratio of 1.5 is associated with an annual default probability of 3% (i.e. a 5-year default probability of roughly

[41] For theoretical analysis of covenants see Smith and Warner (1979), Donaldson et al. (2019, 2020), and references therein.

Variable	Value	Description
r	0.0765	Discount rate of cash flows on long-term real risk-free rate, measured as the 10-year nominal Treasury rate (from FRED) minus 1-year Survey of Professional Forecasters inflation expectations for the GDP deflator. In addition, since we do not have exit in our model (but empirical investment rates would reflect the exit probability) we add the estimated 4.1% exogenous exit rate estimated in in Sterk et al. (2021).
λ	0.3	Use investment spikes literature estimates in Gourio and Kashyap (2007) (Table 1 of the NBER version). They find that in close to 30% of US plant-years have an investment spike of 12% or more of total assets. In particular, as a proportion of investment relative to assets, 11.6% invest between 0.12 and 0.2, 8% invest between 0.2 and 0.35, and 8.3% invest more than 0.35)
(μ, σ, ζ)	$(-0.0514, 0.1534, 50.036)$	Jointly solve with Use g^u from (34) with λ, the closed form first-best dynamics $\mathbb{E}\left[d\log Z_t\right] = \frac{\partial}{\partial t}E\left[\log\left(\frac{Z_t}{Z_0}\right)\right] = \mu - \frac{1}{2}\sigma^2 + \lambda\log\left(1+g^u\right)$ and $\mathbb{V}\left[d\log Z_t\right] \equiv \mathbb{E}\left[\left(d\log Z_t\right)^2\right] - \mathbb{E}\left[d\log Z_t\right]^2 = \sigma^2 + \lambda\log\left(1+g^u\right)^2$ using targets in Sterk et al. (2021)

Table 1: Calibration

15%), that 30% of creditors had an interest coverage ratio of 2 or less, and about 10% of borrowers had an interest coverage ratio of 1 or less. We therefore consider $\ell(0) \in \{3, 9\}$ corresponding to interest coverage ratios of 4 and 1.5 to capture a highly levered and an average firm.