# Commitment and Investment Distortions Under Limited Liability ${ }^{\ddagger}$ 

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We study how frictions originating from the presence of limited liability distort firms' investment and financing choices. By financing new investments with debt, firms can use limited liability to credibly commit to defaulting earlierallowing both firms' owners and new creditors to benefit from diluting existing creditors. In a dynamic setup, this leads to a time-inconsistency, which increases the cost of external funds, and discourage investment. We show that the interaction of these two forces leads to heterogeneous investment distortions where highly-indebted firms overinvest and those with low levels of debt underinvest. Allowing direct payments to firms' owners financed with debt can mitigate overinvestment but, in the presence of repeated investment opportunities, tend to exacerbate the underinvestment of low-leverage firms.
Keywords: limited liability, financial friction, investment, debt financing

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## 1 Introduction

Does investment flow to its most efficient uses when firms are burdened with high debt? While existing debt does not distort investment, financing, or payouts decisions in the absence of market imperfections, a large and diverse literature points to the empirical and theoretical role of financial frictions. This paper develops a model where limited liability, without any additional frictions, lead to distortions in real investment by changing the timing of default.

An essential, but often implicit, feature of most financial frictions is an interaction with limited liability. ${ }^{1}$ For example, in models of private information (e.g., Bernanke et al. (1999) or Clementi and Hopenhayn (2006)) the limited liability of firm owners constrains the possible punishments for misreporting. Similarly, in models of inalienable human capital (Kiyotaki and Moore (1997), and Albuquerque and Hopenhayn (2004)) limited liability distorts investment because the lender cannot operate the firm in default, and the equity holder cannot commit to maintain operations. Models of risk-shifting and entrepreneurship (e.g., Jensen and Meckling (1976) or Vereshchagina and Hopenhayn (2009)) highlight the incentives for risk-taking due to the asymmetric payoff structure implied by limited liability. Finally, in models of limited enforcement (Buera et al. (2011), Moll (2014), among many others) firms' can abscond with profits or capital at the time of default and limited liability prevents effective punishments.

Rather than examine a particular version of these important mechanisms, we instead show that the common element between all of these models-limited liability -alone leads to financial frictions with rich implications for real investment, leverage, and equity payouts. Our emphasis on limited liability as the main friction is driven by the observation that it is a universal institutional feature. Thus, by investigating this friction alone, we identify a common set of distortions faced by all firms in the economy compared to distortions implied by more elaborate models of financial frictions, which may be more individually appropriate for subsets of firms (see Lian and Ma (2021)). In our analysis we consider firm owners' financing options and allow for variable investment and equity payouts (such as dividends or equity buybacks). These ingredients prove to be crucial for our results and

[^1]lead to non-monotone and heterogeneous impacts of outstanding debt on real investment, and-unique among these models of real financial frictions-equity payouts.

To isolate the role of limited liability, we start with a simple baseline model with a single financing-investment opportunity and three types of agents. In our model, firm owners' (i.e., equity holders) operate the firm and are protected by limited liability, preexisting debt investors hold existing debt but are otherwise inactive, and new debt investors competitively price new debt issued by the firm. Firm owners simultaneously choose how much to invest, whether to finance this new investment with debt or their own funds, and how much to directly pay out to themselves. New debt investors price based on this policy, including taking into account that some of the funds may be directly transferred to the firms' ownersthrough equity buybacks or direct dividends - rather than into firm assets. Cash flows are assumed to evolve according to a continuous stochastic process and firm owners can walk away from the firm at any time, in which case debt holders claim the remaining assets. As a result, the optimal default decision is characterized by a threshold on cash flows as in Leland (1994).

We show that with a single investment opportunity, firms have an incentive to invest more than the first-best and finance their investment fully with debt. This incentive to overinvest arises due to "double-selling" of promised earlier cash flows. By financing new investment with debt, firm owners increase leverage and commit to defaulting earlier. This transforms some of the coupon payments promised to the existing debt holders into new bankruptcy claims, which can be partially sold to new debt holders. The incentive to dilute preexisting debt holders' coupon claims increases firm owners' marginal benefit of debt financing, leading to overinvestment and, if permitted, direct equity payouts. While this notion of indirect dilution via changes in the default timing appears in DeMarzo and He (2021), in their setup indirect dilution does not itself provide a force for the firm owners to directly benefit from it. In our model, we show how a mechanism through which firm owners can benefit from this channel can arise, and explain the central role of commitment and time-consistency. Differently from DeMarzo and He (2021) our focus is also on distortion to real investment rather to leverage dynamics. ${ }^{2}$

In the case with no future financing opportunities, a policy of allowing payouts to firm

[^2]owners financed fully with debt is unambiguously socially desirable because it mitigates inefficient overinvestment. ${ }^{3}$ This is because equity payouts provide a more efficient way to increase firm's indebtedness. Thus, all firms, irrespective of their indebtedness, limit their inefficient investment by switching to debt-financed equity payouts. In the corner case with unconstrained equity payouts we can split the firm owners' problem into two separate problems: (1) investment and (2) dilution of existing debt holders. Firm owners choose investment to maximize the net present value of the firm and then choose the level and financing of equity payouts to optimally dilute existing debt holders. Allowing firms to hold cash does not mitigate these distortions since firm owners would never choose to have positive cash holdings. The reason is that while firm owners can still dilute existing claims to coupons, there is no way for firm owners to benefit from that dilution when the proceeds are invested in cash (since the cash is dispersed to debt holders in bankruptcy).

We next incorporate repeated financing-investment opportunities into our model, revealing the importance of dynamic considerations. We model repeated financing-investment opportunities with a fixed stochastic arrival rate. When the arrival rate is non-zero, a new channel emerges as debt investors anticipate firm owners' lack of commitment not to dilute their coupons in the future by changing the default timing, which raises the cost of debt financing at the time of investment. The resulting incentive to underinvest - defined as foregoing investments whose marginal returns exceed their marginal costs - is particularly relevant for firms with low initial indebtedness and a high arrival rate of financing opportunities, because these firms have the largest capacity to dilute the coupons of existing debt holders and frequent opportunities to do so. Therefore, we obtain the prediction that firms with low levels of debt tend to underinvest while high indebted firms-just as in the one-shot model-tend to overinvest. This contrasts the predictions based on models with collateral constraints in which financially constrained firms that are protected by limited liability always underinvest (e.g. in Buera (2009), Moll (2014), Khan and Thomas (2013), or Buera et al. (2015)).

We investigate the effects of a policy that restricts direct equity payouts from debt issuance in a calibrated version of the repeated model. We find that restricting equity payouts has new heterogeneous implications for the efficiency of investment across firms in our model. By mitigating debt investors' concerns about future dilution, equity payout restrictions lower the cost of debt finance for firms with low levels of debt. This raises these firms' investment, which tends to move their investment closer to the efficient level. In contrast, for highly indebted firms with infrequent financing opportunities, restricting

[^3]equity payouts exacerbates inefficient overinvestment as in the baseline model with a single financing opportunity.

Taken together, our results indicate that firms protected by limited liability have an incentive to overinvest because debt-financed investment allows them to (1) dilute current debt holders' coupon claims by increasing indebtedness and bringing forward bankruptcy; or (2) limit the gains from investment to debt holders from new investment if increasing indebtedness is too costly. At the same time, our model predicts that firms with low levels of debt and with frequent investment opportunities tend to underinvest. This is because these firms have the largest capacity to dilute the coupons of debt holders in the future, which in anticipated by creditors who require high compensation for lending to those firms. Facing a high cost of debt, these firms cut their investment below its efficient level. These mechanisms are strengthened when firm owners can deplete equity by making discretionary equity payouts, such as equity repurchases.

## Literature Review -

Our paper contributes to the large literature that investigates how financial frictions distort firms' investment choices and firms' dynamics. Early contributions to this literature include Cooley and Quadrini (2001), Gomes (2001), and Cooper and Ejarque (2003). These papers analyze whether the presence of financial frictions can explain the dependence of firms' dynamics on size and age and the high correlation between investment and cash flows. More recent contributions investigate how financial frictions, typically arising due to limited liability coupled with limited enforcement, affect aggregate outcomes (see, for example, Buera et al. (2011), Khan and Thomas (2013), Moll (2014) or Kohn et al. (2016)). More directly connected with our paper are Albuquerque and Hopenhayn (2004), Clementi and Hopenhayn (2006), and Clementi et al. (2010) who study how financial frictions arising from combinations of limited liability coupled with other imperfections distort firms' investment and affect their growth and how these frictions can be circumvented by optimal dynamic contract (see also Cao et al. (2019)). In contrast to these papers, we focus on limited liability alone and show that even in the absence of additional market imperfections it leads to financial frictions with rich implications for real investment, financing, and equity payouts.

The force that drives our results is the indirect dilution of existing creditors due to a change in the timing of default induced by issuing new debt. DeMarzo and He (2021) were the first ones to point out that in the presence of limited liability issuing new debt can induce such indirect dilution as opposed to the classical direct dilution of bankruptcy claims as emphasized by Fama and Miller (1972) and others. In particular, they investigate how this form of dilution affects leverage dynamics and debt issuance policy. They show that in the presence of tax benefits of debt, firms' owners have incentives to increase leverage, which
then translates into earlier default. However, firms' owners do not directly benefit from the change in the timing of default with the benefits of increasing leverage accruing only due to tax advantage of debt. Compared to DeMarzo and He (2021), our focus is on the consequences of indirect dilution for real investment, equity payouts, and their financing. We also show that in the presence of a positive recovery value, a change in the timing of default can directly benefit firm owners. Thus, our paper contributes to a better understanding of the indirect dilution channel. ${ }^{4}$

Our mechanism is also related to debt dilution and debt overhang in the sovereign default literature (see, for example, Arellano and Ramanarayanan (2012), Chatterjee and Eyigungor (2013), Hatchondo et al. (2016), and Aguiar et al. (2019)). The key difference is institutional: sovereigns have no obligation to pay their creditors anything in default; thus, they dilute the existing debt holders by defaulting earlier and re-capturing payments promised to debt holders. Moreover, the incentives to borrow are different for firms and sovereigns' with firms mostly borrowing to finance investment (which increases future revenues) while sovereigns borrow to mostly finance current expenditure. This means that existing debt distorts the incentive to borrow via different channels.

We also speak to recent empirical and quantitative literature that investigates investment and corporate leverage at the macroeconomic level (e.g. Atkeson et al. (2017)). Recently, Crouzet and Tourre (2020) investigate how business credit programs can mitigate underinvestment and Acharya and Plantin (2019) argue in a model of agency frictions that equity payouts can lead to underinvestment. Finally Jungherr and Schott (2021) and KalemliÖzcan et al. (2022) show how high leverage can lead to slow recovery. We differ from these papers in that we emphasize heterogeneous effects across the firm leverage distribution, where overinvestment is possible and equity payout restrictions may be efficient for some firms but not for others. Finally, our paper is related to Hackbarth and Mauer (2012) who show that debt financing may lead to earlier execution of investment than is prescribed by the first-best solution with this behavior primarily driven by the traditional source of dilution.

[^4]
## 2 Model

There are three types of agents in the baseline model: firm owners (hereafter equity holders) that operate the firm, existing creditors (i.e., debt holders) who hold debt issued in the past, and competitive outside creditors. Equity holders face a one-time opportunity at time 0 , at which time they can issue new debt and equity, make direct payouts to themselves (i.e., dividends or equity buybacks), and make a real investment. Debt is senior to equity but all debt, including newly issued debt, has equal priority. ${ }^{5}$ All actions are perfectly observable and there is complete information. To keep the model analytically tractable and to highlight the underlying intuition, in our baseline model we assume that equity holders have one-time opportunity to raise new financing and make a real investment. We extend the model to feature repeated financing and investment in Section 4.

### 2.1 Firm State, Notation, and Laws of Motion

The state of a firm is summarized by its cash flows, $Z$, and the book value of its liabilities, $L$, defined as the present discounted value (PDV) of all promised cash flows to debt holders. ${ }^{6}$ Equity and bond holders discount future payoffs at the same constant rate $r>0$. In the absence of new investment, $Z(t)$ follows a geometric Brownian motion with risk-neutral drift $\mu$ and instantaneous volatility of $\sigma^{2}>0$

$$
\begin{equation*}
\mathrm{d} Z(t)=\mu Z(t) \mathrm{d} t+\sigma Z(t) \mathrm{d} \mathbb{W}(t), \quad Z(0)>0, \tag{1}
\end{equation*}
$$

where $\mathbb{W}(t)$ is a standard Brownian motion. Liabilities, $L$, may have a one-time jump at time 0 (i.e., at the time of investment if equity holders decide to finance some of the investment with debt) but remain constant for all $t>0$. Thus, we do not allow equity holders to issue new debt or repurchase existing debt after time 0-and relax this assumption in Section 4.

Real Investment and Payouts to Equity At time 0 equity holders have a one-time financing-investment opportunity that expires immediately if not executed. In particular, at time 0 equity holders can deterministically increase initial cash flows of the firm from

[^5]$Z(0)$ to $\hat{Z} \equiv(1+g) Z(0)$, where $g \geq 0$ captures equity holders' investment financed through a combination of new debt and equity. After the initial jump in cash flows, cash flows follow (1). Investment is costly, with its cost being proportional to $Z$ and given by $q(g) Z \equiv \frac{\zeta g^{2}}{2} Z$.

At the time of the investment, we also allow equity holders to make direct payouts to themselves, $M$. We interpret $M$ as equity buybacks, dividends, or leveraged buyouts financed by issuing new debt. Thus, $M$ captures any payout to equity holders that equity holders can use immediately for consumption. The presence of $M$ allows us to consider proposals to limit share buybacks or dividend payments. We assume that equity holders can only consume $M \in[0, \kappa Z]$, where $\kappa \geq 0$ is the parameter capturing institutional constraints on financing equity buybacks with new debt, with $\kappa=0$ being our baseline. ${ }^{7}$

The investment in our model should be interpret as large investment expenditures in the spirit of Gourio (2014). Infrequent adjustments in debt level is also consistent with findings of Welch (2004).

Financing and Limited Liability For tractability we assume that all debt takes the form of defaultable consols, which pay one coupon until the firm defaults and represent a proportional claim to the assets of the firm in bankruptcy. ${ }^{8}$ Equity holders can fund the total cost of investment, $q(g) Z$, and the total equity payouts, $M$, with their own funds (equity financing), by issuing new debt (debt financing), or any linear combination of them. Equity holders are assumed to be deep-pocketed and hence able to finance investment or equity payouts with their own funds if they so choose. We denote the proportion of debt financing by $\psi \in[0,1]$. If the firm issues only equity (i.e., $\psi=0$ ) then the liabilities of the firm, $L$, have no jump at time 0 . If $\psi>0$, liabilities jump at time 0 . In that case, let $\hat{L}$ denote post-investment liabilities - capturing the present value of all coupons.

The post-investment liabilities, $\hat{L}$, is implicitly determined by the financing choice, $\{g, \psi, M\}$, and an equilibrium price, $P(\cdot)$, for any newly issued bonds. We defer details until we can fully specify the problem given the equilibrium actions of equity holders (see Section 2.3).

Equity holders are protected by limited liability. This means that after investment and financing choices have been made, equity holders can choose to default and walk away with

[^6]nothing at any time, whereupon the firm is taken over by debt holders. Equity holders are deep-pocketed, that is they have sufficient funds to keep the firm as a going concern, if they so choose, even when promised debt payments exceed firm cash flows.

Issuing debt with limited liability serves as a mechanism to commit to earlier default. In a repeated version of this model, this leads to time-inconsistency where equity holders would prefer to commit to change the timing of default.

Equity Value At any point in time there are two states variables: current cash flows, $Z$ and current liabilities, $L$. Let $V(Z, L)$ denote the post-investment value of equity (i.e., the value of operating the firm after the investment option was executed) when the current cash flows are $Z$ and current liabilities are $L$. Similarly, let $V^{*}(Z, L)$ denote the pre-investment value of equity (i.e., the value of operating the firm to equity holders at the time they make their investment decision) when time 0 cash flows are $Z$ and time 0 liabilities are $L{ }^{9}$

It will prove useful to rescale the value of equity with cash flows. Thus, we denote the post- and pre-investment equity value relative to cash flows as $v(\cdot) \equiv V(\cdot) / Z$ and $v^{*}(\cdot) \equiv V^{*}(\cdot) / Z$, respectively. Similarly, we define current leverage as $\ell \equiv L / Z$ and the equity payouts per unit of $Z$ as $m \equiv M / Z$.

Value in Default Upon default the firm is taken over by the debt holders who continue to operate it. However, default may have real costs in the sense that immediately following default the firm's cash flows decrease from $Z$ to $(1-\theta) Z$, where $\theta \in[0,1]$. The parameter $\theta$ captures deadweight costs associated with bankruptcy proceedings and debt holders' lower skill in running the firm. We use $\theta=0$ as our baseline but show robustness to empiricallyplausible alternative values for $\theta$.

### 2.2 Equity Holders' Investment and Default Decisions

Equity holders face the following decisions. First, at time 0, they have to choose how much to invest, $g$, how much to pay out to themselves, $M$, and how to finance these choices, $\psi$. Having made these choices, at each instant of time they need to decide whether to keep operating the firm or default instead.

Investment Decision Given an initial state $(Z, L)$, equity holders choose the financing mix $\psi$ and real investment $g$ to maximize the sum of the post-investment equity value and direct payouts to equity, net of new equity injected into the firm. The post-investment

[^7]equity value is given by $V((1+g) Z, \hat{L})$, where $\hat{L}$ are the post-investment liabilities. Equity holders take the equilibrium price $P(\cdot)$ for newly issued bonds as given, and solve
\[

$$
\begin{align*}
& V^{*}(Z, L)= \max _{\substack{g>0 \\
\psi \in[0,1] \\
0 \leq M \leq \kappa Z}}\{\overbrace{V(\underbrace{(1+g) Z}_{=\hat{Z}}, \hat{L})}^{\text {Post-Investment Equity }}-\overbrace{(1-\psi) q(g) Z}^{\text {Equity Financed }}+\overbrace{M}^{\text {Payouts }}\}  \tag{2}\\
& \quad \text { s.t. } \underbrace{P(\hat{L}, Z, L, g, \psi, M)}_{\text {Equilibrium Price }} \underbrace{(r \hat{L}-r L)}_{\text {New Bonds }}=\underbrace{\psi q(g) Z}_{\text {Debt Financed }}+M \tag{3}
\end{align*}
$$
\]

and subject to the feasibility of the payoffs in default embedded in $L$ and $\hat{L}$. In (3), $r(\hat{L}-L)$ is the quantity of new bonds issued (given the constant interest rate $r$ ). Funds raised from issuing new debt at price $P(\cdot)$ can be used for equity payouts, $M$, or to finance a portion of investment costs, $\psi q(g) Z$. To understand how existing liabilities distort equity holders' investment choices we consider the following first-best benchmark.

Definition 1 (First-Best Investment). We define the first-best undistorted investment, $g^{u}$, as investment that maximizes the net present value of the firm. That is,

$$
\begin{equation*}
g^{u}(Z) \equiv \arg \max _{g}\{\overbrace{V((1+g) Z, 0)}^{\text {Post-Investment Equity }}-\overbrace{q(g) Z}^{\text {Equity Financed }}\} \tag{4}
\end{equation*}
$$

Thus, first-best investment is the level that equity holders would choose if the firm had no preexisting debt and investment had to be fully financed with equity. Since both the payoffs and costs are linear in $Z$, we can show that $g^{u}$ is independent of $Z$ throughout our model. The homotheticity that leads to a constant $g^{u}$ is not essential, but simplifies the analysis.

Default Decision Equity holders optimally choose to default when the equity value, $V(Z, L)$, reaches 0 . Note that after investment only the cash flows fluctuate, and the equity holders' continuation problem becomes a standard stopping problem (as in Leland (1994) with liabilities $L$ as an additional state), which is given by

$$
\begin{align*}
r V(Z, L) & =Z-r L+\mu Z \boldsymbol{\partial}_{Z} V(Z, L)+\frac{\sigma^{2}}{2} Z^{2} \boldsymbol{\partial}_{Z Z} V(Z, L)  \tag{5}\\
V(\underline{Z}, L) & =0  \tag{6}\\
\boldsymbol{\partial}_{Z} V(\underline{Z}, L) & =0 \tag{7}
\end{align*}
$$

where $\underline{Z}$ is the endogenous default barrier. Here (6) and (7) are the standard value-matching and smooth pasting conditions, respectively. Define

$$
\begin{align*}
\eta & \equiv \frac{\left(\mu-\sigma^{2} / 2\right)+\sqrt{\left(\mu-\sigma^{2} / 2\right)^{2}+2 \sigma^{2} r}}{\sigma^{2}}>0  \tag{8}\\
\chi & \equiv\left(\frac{(r-\mu) \eta}{\eta+1}\right)^{\eta}>0  \tag{9}\\
s(\ell) & \equiv \frac{\chi}{\eta+1} \ell^{\eta}, \tag{10}
\end{align*}
$$

where $s(\ell)$ captures the value of equity holders' option to default per unit of liabilities (see the discussion below). We now characterize the solution to equity holders' default problem (5)-(7).

Proposition 1 (Post-Investment Equity and Default). Suppose that the current state of the firms is $(Z, L)$. Then the equity value of the firm is given $V(Z, L)=v(\ell) Z$, where $\ell \equiv L / Z$ is firm's leverage and

$$
\begin{equation*}
v(\ell)=\frac{1}{r-\mu}-\ell(1-s(\ell)) \tag{11}
\end{equation*}
$$

The endogenous default threshold $\underline{Z}$ satisfies

$$
\begin{equation*}
\frac{\underline{Z}(L)}{r-\mu}=\frac{\eta}{1+\eta} L \tag{12}
\end{equation*}
$$

and the liquidation value per unit of liabilities is given by

$$
\begin{equation*}
\frac{V((1-\theta) \underline{Z}(L), 0)}{L}=\frac{(1-\theta) \eta}{1+\eta} \tag{13}
\end{equation*}
$$

Proof. See Appendix A.1.
The above proposition characterizes the value of equity and equity holders' optimal default decision. Note that if equity holders were not allowed to default then the value of equity would be given by $\left(\frac{1}{r-\mu}-\ell\right) Z$ as equity holders would have to repay all their liabilities. Thus, $s(\ell) L$ captures the value of equity holders' option to default. In what follows, we assume that $Z_{0}>\underline{Z}$.

Before moving on to how debt is priced, it is useful to visualize how firm cash flows are divided between equity and debt holders. Figure 1 depicts a possible path of $Z$ and shows how cash flows are divided among all claimants (for simplicity we set $\theta=0$ ). Default occurs at the threshold $\underline{Z}(L)$ optimally chosen by the firm, following Proposition 1. Because we are considering the baseline case without bankruptcy costs $(\theta=0)$, firm's cash flows are


Figure 1: The effect of limited liability on debt and equity cash flows without investment. This figure considers the simplified case with no bankruptcy costs $(\theta=0)$.
not impacted by the default event-instead they simply change claimants. Prior to default, coupons are paid and residual funds are distributed to equity holders. After default, all cash flows are owned by the default claimants. The price of any financial claim is simply the expected present discounted value of cash flows allocated to this claimant, with the expectation taken over all possible realizations of the $Z(t)$ process.

### 2.3 Pricing of Debt

In this section, we derive how debt is priced, which then determines the equilibrium budget constraint in (3). Debt is priced by outside creditors who are risk-neutral and who anticipate equity holders' optimal default decision (as characterized in Proposition 1). Let $T$ denote the stopping time when cash flows first reach default threshold, $\underline{Z}$, at which point equity holders choose to default.

For tractability, we assume that all debt takes the form of defaultable consols following Leland (1994, 1998). A defaultable consol pays 1 in perpetuity prior to default and receives a share of the bankruptcy value of the firm in default. Conditional on the current state of the firm $(Z, L)$ and equity holders' optimal default decision, the market price of a such bond $P(Z, L)$ equals

$$
P(Z, L) \equiv \frac{p(Z, L)}{r}=\underbrace{\mathbb{E}_{T}\left[\int_{0}^{T} e^{-r \tau} \mathrm{~d} \tau\right]}_{\begin{array}{c}
\text { PDV of promised coupons }  \tag{14}\\
=P^{C}(Z, L)
\end{array}}+\underbrace{\mathbb{E}_{T}\left[e^{-r T} \frac{V((1-\theta) \underline{Z}(L), 0)}{r L}\right]}_{\begin{array}{c}
\text { PDV of claims in bankruptcy } \\
=P^{B}(Z, L)
\end{array}},
$$

where $p(Z, L)$ denotes the price relative to the risk-free rate.
Equation (14) emphasizes that a defaultable consol consists of pre-bankruptcy com-
ponent (i.e., the coupon payments prior to default) with market price $P^{C}(Z, L)$, and a component consisting of claims in bankruptcy (i.e., all debt is equal priority or pari passu) with market price $P^{B}(Z, L)$. We use $p^{C}(\cdot, \cdot)$ and $p^{B}(\cdot, \cdot)$ to denote these prices relative to the risk-free rate, $r$.

Proposition 2 establishes that leverage $(\ell \equiv L / Z)$ is the relevant state for pricing debt, solves for the prices of the defaultable consol and of its pre-bankruptcy and bankruptcy components, and characterizes equity holders' budget constraint, (3).

Proposition 2 (Pricing of Debt). The relevant state for pricing debt is leverage $\ell \equiv L / Z$. Moreover,

$$
\begin{align*}
p(\ell) & =1-(1+\theta \eta) s(\ell)  \tag{15}\\
p^{C}(\ell) & =1-(1+\eta) s(\ell)  \tag{16}\\
p^{B}(\ell) & =(1-\theta) \eta s(\ell) \tag{17}
\end{align*}
$$

Given these prices, the budget constraint (3), normalized by $Z$, is given by

$$
\begin{align*}
p(\hat{\ell})((1+g) \hat{\ell}-\ell) & =\psi q(g)+m  \tag{18}\\
p(\hat{\ell}) & \geq p^{B}(\hat{\ell}) \tag{19}
\end{align*}
$$

where $\hat{\ell} \equiv \hat{L} / \hat{Z}$ is post-investment leverage.

Proof. See Appendix A.2.
The normalized budget constraint (18) is standard. The left hand side equals the total value of new debt issued, normalized by $Z$. The right hand side represents equity holders' need to raise new debt financing, and equals the debt financed portion of investment costs plus equity payouts, also normalized by $Z$. The constraint (19) enforces the feasibility of the default payoff embedded in defaultable consols, meaning that equity holders cannot issue so much debt that the firm is strictly above its default threshold immediately after investment. While the constraint (19) never binds when $\kappa=0$, it may bind if direct payments to equity holders are allowed $(\kappa>0) .{ }^{10}$

In the case without bankruptcy costs (i.e., $\theta=0$ ) the price of the defaultable consol relative to the risk-free rate simplifies to $p(\ell)=1-s(\ell)$. In this particularly simple case, $s(\ell)$ can be interpreted as the spread relative to the risk-free rate. Recall that $s(\ell)$ also

[^8]appears in the expression for $v(\ell)$ (the normalized value of equity) with the opposite sign, where it captures the value of equity holders' option to default. Thus, we see that a more valuable default option increases the value of equity at the expense of bond holders.

Bankruptcy claims Proposition 2 prices an asset which bundles coupon payments with a claim to a fraction of the liquidation value of the firm upon default (i.e., assuming the proportional distribution of claims is usually referred to as pari passu). A concern might be that since there is no seniority of existing claims on the firm in bankruptcy relative to new claims, the firm is able to capitalize on that by issuing so much debt (possibly an infinite quantity) to dilute existing debt holders' bankruptcy claims (see, for example, the discussion in Section 2.D of DeMarzo and He (2021)).

This, however, is not possible in our model. First of all, constraint (19) restricts equity holders to issue debt up to a finite limit implied by the default threshold. Second, for all $\ell$ that satisfy constraint (19), the value of bankruptcy claims, $p^{B}(\ell)$ is actually increasing in $\ell$. The intuition is that while new debt does increase the number of claimants in bankruptcy, it also induces firms to default with higher $Z$. This increases the liquidation value of the firm (as the liquidation value is proportional to $L$ in (13)) and offsets the effect of an increase in the number of claimants in bankruptcy. In addition, since the firm now defaults on average earlier, the present value of existing debt holders' bankruptcy claims actually increases.

### 2.4 Collateralized Debt as an Equivalent Formulation

We can equivalently assume that the existing debt is collateralized to the firm's preinvestment assets, without any changes to the equity holders' optimal choices. Since the value of the firm in bankruptcy is known with certainty, we can interpret that value as collateral that equity holders' can use to collateralize debt. We then impose the constraint on equity holders' so that any collateral pledged to existing debt holders cannot be pledged to new debt holders (or more generally, that equity holders' cannot pledge the same collateral to multiple debt holders). Let us denote by $p^{S}(\cdot)$ the value of existing collateralized debt relative to risk-free interest rate, $r$. Similarly, let $\hat{p}^{S}(\cdot)$ denote the new collateralized debt sold to finance investment.

As we show in Appendix A.2, in equilibrium, we have $p^{S}(\hat{\ell})=\hat{p}^{S}(\hat{\ell})=p(\hat{\ell})$. That is, the equilibrium price of the existing and new collateralized debt is identical, and both are the same as the price of the baseline asset. Moreover, we show that the (19) is implied by the pledgeability constraint that ensures that equity holders' cannot pledge the same collateral to multiple debt holders. Thus, we conclude that equity holders will make the same real investment decisions when debt is collateralized-highlighting a key distinction between our model and financial frictions related to limited enforcement around collateral
in bankruptcy such as in Buera et al. (2011) and Moll (2014).

### 2.5 Equity Holders' Investment Problem and the First-Best

We now restate equity holders' investment problem, normalizing by current cash flows and substituting in the budget constraint associated with defaultable consols (18). Equity holders of a firm with pre-investment leverage $\ell$ choose $(g, m, \psi, \hat{\ell})$ such that

$$
\begin{align*}
& v^{*}(\ell)= \max _{\substack{, \hat{\ell} \geq 0 \\
\psi \in[0,1] \\
0 \leq m \leq \kappa}}\{\overbrace{(1+g) v(\hat{\ell})}^{\text {Post-Investment Equity }}-\overbrace{(1-\psi) q(g)}^{\text {Equity Financed }}+\overbrace{m}^{\text {Payouts }}\}  \tag{20}\\
& \text { s.t. } \underbrace{p(\hat{\ell})}_{\begin{array}{c}
\text { Bond Price } \\
p(\hat{\ell}) \geq p^{B}(\hat{\ell})
\end{array}} \underbrace{((1+g) \hat{\ell}-\ell)}_{\text {New Bonds }}=\underbrace{\psi q(g)}_{\text {Debt Financed }}+\underbrace{m}_{\text {Payouts }} \tag{21}
\end{align*}
$$

The first-best investment from Definition 1 solves

$$
\begin{equation*}
g^{u} \equiv \arg \max _{g}\{\overbrace{(1+g) v(0)}^{\text {Post-Investment Equity }}-\overbrace{q(g)}^{\text {Equity Financed }}\} \tag{23}
\end{equation*}
$$

The equity holders' objective function (20) is a normalization of (2). Similarly, the first-best investment in (23) is just the normalization of (4). As noted in Section 2.3, the constraint (22) enforces the feasibility of the default payoff embedded in defaultable consols, meaning that equity holders cannot issue so much debt that the firm is strictly above its default threshold immediately after investment.

## 3 Analysis of Investment Decision

In this section, we analyze equity holders' investment and financing decisions in our baseline model with a single financing-investment opportunity. We first characterize the equity holders' problem relative to the first-best. This comparison allows us to identify the sources of inefficiencies in equity holders' investment decisions.

### 3.1 Sources of Investment Distortions

We define the function $H(\hat{\ell}) \equiv \theta \eta s(\hat{\ell})$ as the deadweight cost of default per unit of leverage, which is non-zero if a share of the firm is dissipated in default (i.e., if $\theta>0$ ). We then characterize the investment problem as follows.

Proposition 3. Equity holders' investment problem can be written as

$$
\begin{align*}
& v^{*}(\ell)=\max _{\substack{g, \hat{\ell} \geq 0 \\
\psi \in 0,1] \\
0 \leq m \leq \kappa}}\left\{\frac{1+g}{r-\mu}-q(g)-p(\hat{\ell}) \ell-(1+g) H(\hat{\ell}) \hat{\ell}\right\}  \tag{24}\\
& \quad \text { s.t. } p(\hat{\ell})((1+g) \hat{\ell}-\ell)=\psi q(g)+m  \tag{25}\\
& p(\hat{\ell}) \geq p^{B}(\hat{\ell}) \tag{26}
\end{align*}
$$

The first-best investment, $g^{u}$, is the unique solution to

$$
\begin{equation*}
0=\frac{1}{r-\mu}-q^{\prime}\left(g^{u}\right) \tag{27}
\end{equation*}
$$

If the investment cost is quadratic and equal to $q(g)=\zeta g^{2} / 2$ then the optimal investment is given by $g^{u}=\frac{1}{\zeta(r-\mu)}$.

Proof. See Appendix A. 3 for details.
The reformulated objective function (24) shows that the post-investment value of equity equals the expected PDV of cash flows generated by the firm net of (i) the cost of investment $(q(g))$, (ii) the PDV of cash flows promised to the existing debt holders $(p(\hat{\ell}) \ell)$, and (iii) cash flows lost in default $\left((1+g) H(\hat{\ell} \hat{\ell})\right.$, all normalized by $Z .{ }^{11}$ Because new debt is fairly priced equity holders bear the full cost of the investment and any change in the expected deadweight cost of default. For the same reason equity payouts $m$ do not appear directly in (24). However, $\psi$ and $m$ affect the post-investment value of equity indirectly through $\hat{\ell}$.

The above characterization of the equity holders' problem emphasizes the sources of inefficient investment. Compared to (23), we see that equity holders face two distortions. The first distortion is due to existing debt, as captured by $p(\hat{\ell}) \ell$. Since $p(\hat{\ell})$ is a decreasing function of post-investment leverage, $\hat{\ell}$, equity holders have an incentive to increase leverage. This is the classic conflict between equity and debt holders pointed out by Myers (1977)though in contrast to our model, Myers (1977) assumes that investment is entirely financed with equity. The second distortion is due to bankruptcy costs and is captured by $(1+$ g) $H(\hat{\ell}) \hat{\ell}$. Since $H(\cdot)$ is an increasing function, the presence of bankruptcy costs discourages equity holders from taking on additional leverage.

[^9]
### 3.2 Investment Relative to First-Best

We focus first on the case without bankruptcy costs (i.e., $\theta=0$ ), so the last term in (24) vanishes, and preexisting leverage is the only source of investment distortions. ${ }^{12}$ If in addition preexisting debt is zero (i.e., $\ell=0$ ), it follows from the reformulated investment problem in Proposition 3 that equity holders' optimal investment satisfies

$$
\begin{equation*}
\frac{1}{r-\mu}-q^{\prime}(g)=0 \tag{28}
\end{equation*}
$$

that is investment equals the first-best $\left(g=g^{u}\right)$. Equity holders are also indifferent between equity and debt financing and whether to make equity payouts (i.e., they are indifferent over any feasible choices of $\psi$ and $m$ ). In other words, when $\ell=0$ the Modigliani-Miller theorem holds, and the firm value is independent of financing.

Investment deviates from this simple benchmark when the firm has preexisting debt $(\ell>0)$. Without bankruptcy costs, it is immediate from (24) that equity holders' optimal investment satisfies the following first-order condition (FOC)

$$
\begin{equation*}
\frac{1}{r-\mu}-q^{\prime}(g)-p^{\prime}(\hat{\ell}) \frac{\partial \hat{\ell}}{\partial g} \ell=0 \tag{29}
\end{equation*}
$$

Equation (29) is the key equation of our model. ${ }^{13}$ Compared to (28), which determines equity holders' investment choice when $\ell=0$, the FOC now includes the additional term $-p^{\prime}(\hat{\ell}) \frac{\partial \hat{\ell}}{\partial g} \ell$. This new term captures the marginal change in the value of existing debt due to the change in the firm's distance to default. When this term is positive at $g=g^{u}$ equity holders have an incentive to invest beyond the level that would maximize the total value of the firm, while the opposite is true when the term is negative. Since the value of existing debt is decreasing in leverage (i.e., $p^{\prime}(\hat{\ell})<0$ ) it follows that the sign of this distortion depends on the sign of $\frac{\partial \hat{l}}{\partial g}$. If at optimal investment we have $\frac{\partial \hat{l}}{\partial g}>0$ then equity holders overinvest relative to first-best. If at the optimal investment we have $\frac{\partial \hat{\ell}}{\partial g}<0$ then equity holders underinvest.

### 3.3 Dilution Mechanism and Inefficient Investment

We now present our first main result that preexisting debt encourages overinvestment. We first characterize equity holders' choices without equity payouts financed by debt (i.e.,

[^10]$\kappa=0) .{ }^{14}$
Proposition 4. Given $\kappa=0$, denote $g^{*}$ as equity holders' optimal investment

1. If constrained to use equity financing, equity holders underinvest ( $g^{*}<g^{u}$ )
2. If allowed to choose financing optimally, equity holders: (a) finance all their investment with debt; (b) overinvest $\left(g^{*}>g^{u}\right)$

Proof. See Appendix B. 2


Figure 2: Equity-financed investment, due to deleveraging, decreases the option value of default. This figure shows debt and equity cash flows with an equity-financed investment opportunity. We show the simplified case with no bankruptcy $\operatorname{costs}(\theta=0)$.

The first part of Proposition 4 nests the classic underinvestment result of the debt overhang literature (Myers (1977)). Thus, our model makes precise that equity financing is a condition required for this classic result. Figure 2 visualizes the intuition. Investment financed with equity leads to deleveraging, leading equity holders to pay coupons to creditors for longer. As a result, a portion of the cash flows from the new investment is allocated to existing debt holders in form of coupon payments (the portion of the "Claims in Default $\rightarrow$ Creditors" area above the dotted line). The benefit from new investment is hence partly captured by existing debt holders, implying that equity holders' benefit from investment is lower than the social benefit. In terms of the key equation (29), deleveraging implies that $-p^{\prime}(\hat{\ell}) \frac{\partial \hat{\ell}}{\partial g} \ell<0$, and hence a reduction of equity holder's incentive to invest. This classic argument has been used to explain the historically low investment in the aftermath of the Great Recession in Europe (see, for example, Kalemli-Özcan et al. (2022)). ${ }^{15}$

[^11]By contrast, the second part of Proposition 4 shows that equity holders may want to overinvest if they can choose their financing method freely, provided that the firm has preexisting debt (i.e., $\ell>0$ ). Note that investment is deterministic and the volatility of firm's cash flows is assumed to remain the same before and after investment, and thus the mechanism here is distinct from the potential incentive to invest in risky projects with negative present value, so-called risk-shifting (Jensen and Meckling (1976)).


Figure 3: Debt-financed investment, due to increased leverage, dilutes existing debt holders by double-selling some of their promised coupon payments. This figure shows debt and equity cash flows with a debt financed investment opportunity. We show the simplified case with no bankruptcy costs $(\theta=0)$.

Why do equity holders overinvest? This is driven by two related forces: (1) an incentive to dilute existing debtholders (that motivates low- and medium-leverage firms' choices), and (2) the incentive to limit the gain from new investment to existing debtholders (that motivates high-leverage firms' choices, for whom costs of dilution exceed its benefits). As we will explain in detail below, in both cases equity holders issues additional debt (above socially optimal level) to shorten the expected length of time they have to pay coupons to existing debt holders. This is different from the traditional source of dilution (i.e., issuing new equal priority claims to the firm in default) that we deliberately shut down to isolate our new mechanism (see (19) and its discussion).

How equity holders can dilute existing debt holders in the model? Figure 3 illustrates that a sufficiently large debt-financed investment leads to higher leverage and earlier default, transforming a portion of the coupon payments that has been promised to existing debt holders (the rectangular area with dashed edges) to claims in default, which have to be shared with new debt holders. Thus, by issuing new debt and increasing leverage, equity holders can sell again claims to some of the cash flows that were previously promised to existing debt holders. It follows that the marginal benefit to equity holders of investing
exceeds the social benefit, and hence equity holders overinvest relative to the first-best. In terms of (29), this additional benefit of financing investment with debt is captured by $-p^{\prime}(\hat{\ell}) \frac{\partial \hat{\ell}}{\partial g} \ell>0$.

It is worth pointing out that only the coupon payments promised to existing debt holders are double-sold by equity holders, not the existing debt holders' bankruptcy claims. In particular, the value of existing debt holders' bankruptcy claims at the time of default is unaffected by the increase in leverage. ${ }^{16}$ Furthermore, since after investment default happens on average earlier, the PDV of existing bankruptcy claims actually goes up. Thus, the dilution mechanism in our model operates through changes in the default timing and affects negatively only the value of existing debt holders' coupon claims. As such this mechanism differs from the standard dilution mechanism emphasized in the literature, where equity holders issue an excessive amount of debt (particularly just before default) to dilute existing creditors' claims in default (Fama and Miller (1972)).

The similar mechanism drives overinvestment by firms with low and medium leverage. If initial leverage is sufficiently high equity holders choose to deleverage (i.e., choose $\hat{\ell}<\ell$ ). This is because at high levels of initial leverage, increasing leverage further requires very large inefficient investment. The cost of such inefficient investment are borne by the equity holders and exceed the benefits of dilution. ${ }^{17}$ However, even if equity holders deleverage they still overinvest. This is because, on the margin, increasing investment above the efficient level and financing it with debt allows the equity holders to limit the amount of cash flows generated by new investment that would be captured by existing debt holders. This observation further underscores the difference between our mechanism and earlier work on risk-shifting and debt dilution. ${ }^{18}$

The overinvestment incentive identified in Proposition 4 stands in a stark contrast to popular models of financial frictions featuring collateral constraints. In the latter, limited liability coupled with additional sources of market incompleteness (such as private informa-

[^12]tion in Clementi et al. (2010) or the ability to abscond funds in Buera et al. (2011)) leads to underinvestment by constrained firms (i.e., firms with low assets relative to debt or borrowing needs). In contrast, we show that limited liability by itself provides an incentive to overinvest. In the model with one-shot financing-investment opportunity, this leads to overinvestment while in the presence of repeated financing-investment opportunities it results in heterogeneous investment distortions that depend on firms' leverage (see Section 4). ${ }^{19}$

### 3.4 Equity Payouts

Having seen that the availability of debt financing can lead to overinvestment in the presence of preexisting debt, we now turn to analyzing how debt financed payouts to equity holders, such as dividends and equity buybacks, affect real investment (i.e., $\kappa>0$ ). We define $\bar{m}(\psi, g)$ as the highest equity payout that satisfies the budget constraint (21) given investment, $g$, and financing choice, $\psi$. Thus, given $g, \psi$, equity holders' choice of $m$ has to satisfy $m<\min \{\kappa, \bar{m}(\psi, g)\}$. As shown in Appendix B. $3((\mathrm{~B} .7)), \bar{m}(\psi, g)=\frac{1+g}{r-\mu}-\frac{\eta}{1+\eta} \ell-\psi q(g)$.

Proposition 5. Denote $g^{*}, m^{*}, \psi^{*}$ as the equity holders' optimal choices of investment, payouts, and financing, respectively. There exists $\underline{\kappa} \in \mathbb{R}_{+}$such that

1. If $\kappa<\underline{\kappa}$ then equity holders: (a) overinvest ( $g^{*}>g^{u}$ ); (b) finance investment and equity payouts with debt $\left(\psi^{*}=1\right)$; (c) make payouts to themselves up to the constraint ( $m^{*}=\kappa$ ); (d) continue operating firm
2. If $\kappa \geq \underline{\kappa}$ then equity holders: (a) invest the first-best amount $\left(g^{*}=g^{u}\right)$; (b) finance investment and equity payouts at least partially with debt $\left(\psi^{*} \in\left[\max \left\{\underline{\psi}_{\kappa}, 0\right\}, 1\right]\right.$, where $\underline{\psi}_{\kappa}>0$ is the unique solution to $\kappa=\bar{m}\left(g^{*}, \underline{\psi}_{\kappa}\right)$ ); (c) make maximal feasible payouts to themselves $\left(m^{*}=\bar{m}\left(g^{*}, \psi^{*}\right)\right)$; (d) are indifferent between defaulting and continuing to operate the firm

The threshold $\underline{\kappa}$ is decreasing in $\ell$ and $r$, and increasing in $\sigma$.
Proof. See Appendix B.3.
Proposition 5 extends our overinvestment result to the case in which equity payouts financed with debt are permitted. As long as equity holders face sufficiently tight restrictions

[^13]on equity payouts financed by debt $(\kappa<\underline{\kappa})$, we find that they continue to overinvest. Different from the case with $\kappa=0$, equity holders accompany investment with direct equity payouts, further increasing post-investment leverage.

By contrast, when the constraint on equity payouts is $\operatorname{lax}(\kappa \geq \underline{\kappa})$, equity holders invest the first-best amount. In this case, equity holders have a more efficient way of increasing leverage than inefficient investment. Thus, when $\kappa \geq \underline{\kappa}$, equity holders' problem is decoupled into two separate problems: (1) an investment problem and (2) a dilution of existing debt holders problem. Equity holders choose $g$ to maximize the net present value of the firm and $m$ to maximize the transfer from existing debt holders to themselves. The latter implies choosing the highest feasible $m$ so that the firm defaults right after investment. Thus, when $\kappa \geq \underline{\kappa}$, equity holders essentially sell the firm to the new debt holders. ${ }^{20}$

Proposition 5 shows that restrictions on equity payouts can increase investment and reduce the probability of bankruptcy, in line with the intuition in Myers (1977). Different from Myers (1977), however, we find that such restrictions might not be desirable. We reach different conclusion since in our model investment is debt financed and tends to be inefficiently high.

The conclusion that equity payout restrictions lead to inefficiently high investment needs to be tempered by the observation that this one-shot investment problem does not account for potential future debt dilutions. We will see in Section 4 that dynamic considerations lead to more nuanced conclusions. However, the overinvestment incentive highlighted in the one-shot model tends to remain particularly for firms with relatively high leverage.

Finally, one may wonder if covenants could be used to eliminate the inefficiencies characterized in Propositions 4 and 5. However, as has been first pointed by Myers (1977), it is difficult to write and enforce debt contracts which ensure that equity holders invest firstbest amount. Indeed, as we discuss in Section 4.3, commonly used covenants are unlikely to correct equity holders' incentives.

### 3.5 Cash and Investment Timing

Above we showed that by selling new debt and increasing leverage, equity holders can credibly commit to default earlier. This transforms cash flows previously promised as coupon payments to old debt holders into new cash flows associated with the firm in liquidation. Equity holders benefit from increasing leverage by either making inefficiently high investments and pocketing the excess cash flows prior to default or by directly making payouts to themselves (if $\kappa>0$ ). This emphasis on timing leads to some important questions on

[^14]robustness: Would the investment distortions remain if the firm could hold cash? And if it could hold cash, is it crucial that the timing of investment coincides with the timing of raising funds from the capital markets?

As we show in Appendix B.6, our mechanism is robust to introducing cash into the model since equity holders have no incentive to hold cash inside the firm. Intuitively, a firm which issues debt and saves the proceeds in form of cash continues to have the ability to dilute existing claims to coupons as in Section 3.3. However, the difference is that in this case there is no way to for equity holders to directly benefit from this dilution as cash is added to the assets of the firm and is distributed to debt holders in bankruptcy. The indirect benefit itself is not enough to compensate the equity holders for the cost of raising cash and, thus, equity holders have no incentive to raise cash. This result indicates that if the firm needed to hold cash for other reasons (e.g., as liquidity while waiting for new investment opportunities) the possibility that cash will be distributed to debt holders in bankruptcy would deter the firm from holding the optimal amount of cash within the firm.

The above discussion also suggests that the assumption that financial and investment decisions occur simultaneously is not important for our conclusions. The key distortions in our model arise as long as equity holders have some way to directly benefit from diluting existing coupon claims. A simple extension of our model could decouple financing and investment opportunities by modeling them as independent Poisson processes. We conjecture that in this case at each investment opportunity the firm would issue debt to raise cash that would then be invested at the next investment opportunity. We think of the reduced-form cost of real investment in our simpler model as capturing a variety of costs, potentially including the cost of holding cash while waiting for the next investment opportunity, so the main forces leading to over- and under-investment should be similar in such a model extension. This logic suggests that the important assumption on the timing in our model is therefore that opportunities to sell new debt are lumpy.

### 3.6 Bankruptcy Costs

The above analysis abstracts from default costs $(\theta=0)$. In the presence of default costs (i.e., $\theta \in(0,1])$, debt financing is associated with the following trade-off. On the one hand, as before, issuing new debt allows equity holders to resell some of existing debt holders' claims, which we have seen encourages overinvestment. On the other hand, issuing new debt increases the deadweight cost of default and hence the cost of debt financing, encouraging underinvestment.

It is straightforward to show analytically that for sufficiently low $\theta$ Proposition 4 and Proposition 5 continue to hold. This is intuitive: when $\theta$ is small then the benefit of issuing additional debt exceeds the associated increase in the deadweight cost of default.

In contrast, for sufficiently high $\theta$ an increase in deadweight cost of default associated with issuance of new debt dominates. As a consequence, in this case equity holders finance their investment with equity and underinvest. The key question is then which force dominates for empirically plausible bankruptcy costs. Figure 4 shows that for empirically plausible bankruptcy costs that the incentive to double-sell some of the existing debt holders' coupon claims dominates and equity holders overinvest for all levels of leverage,


Figure 4: Investment relative to first-best ( $\tilde{g}$ ) against preexisting leverage ( $\ell$ ) for the baseline model with a single financing-investment opportunity. Each line corresponds to a different level of deadweight bankruptcy costs $(\theta)$. The case $\theta=0$ corresponds to no bankruptcy costs. Model parameters are discussed in Appendix E.

## 4 Repeated Financing and Investment

So far, we have analyzed a model with a single financing-investment opportunity. In that setting, we have shown that equity holders have an incentive to finance their investment with debt and overinvest. We have argued that this behavior is driven by the incentive to dilute preexisting debt holders coupons (among low leverage firms) and to limit the gain from new investment to existing debt holders (among high-leverage firms). In this section, we build on this model and allow for repeated financing-investment opportunities. We show that repeated financing has non-trivial consequences as buyers of new debt price in the likelihood of future dilution, thereby increasing the cost of debt financing. We find that dynamic considerations can lead to underinvestment especially among low leverage firms with frequent financing-investment opportunities, and that direct equity payouts (i.e., equity buybacks and dividends) financed out of debt further exacerbate underinvestment
among these firms because they make debt financing more expensive.

### 4.1 Model with Repeated Investment

We consider the same setup as described in Section 2, but assume that financing-investment opportunities arrive at a constant Poisson rate, $\lambda$. As above, the state of the firm at any given point in time is $\{Z, L\}$. Upon arrival of a financing-investment opportunity, equity holders have the choice to increase current cash flows from $Z$ to $Z(1+g)$ at cost $Z q(g)$. It follows that cash flows follow a jump diffusion

$$
\begin{equation*}
\mathrm{d} Z(t)=\mu Z(t) \mathrm{d} t+\sigma Z(t) \mathrm{d} \mathbb{W}(t)+g\left(Z\left(t^{-}\right), L\left(t^{-}\right)\right) Z\left(t^{-}\right) d \mathbb{N}(t), \quad Z(0)>0, \tag{30}
\end{equation*}
$$

where $\mathbb{N}(t)$ is a Poisson process with intensity $\lambda \geq 0$ and $g\left(Z\left(t^{-}\right), L\left(t^{-}\right)\right)$is equity holders' investment at time $t$ conditional on the state of the firm $\{Z, L\}$ and the arrival of a financinginvestment opportunity. Note that when $\lambda=0$ we are back to the baseline model with a single financing-investment opportunity.

Investment can be financed by issuing defaultable consols via competitive debt markets (as described in Section 2.3, but with prices reflecting the dynamic decisions of the firm) or equity. This implies that, in contrast to the model of Section 2, liabilities are no longer constant. Rather, $L(t)$ is now a pure jump process with $\mathrm{d} L(t)=\left(\hat{L}(t)-L\left(t^{-}\right)\right) \mathrm{d} \mathbb{N}(t)$, where $\hat{L}(t)$-as defined in Section 2.1-denotes the value of liabilities immediately after investment implied by equity holders' investment and financing decisions.

### 4.2 Optimal Investment Problem

Conditional on the arrival of a financing-investment opportunity, the firm solves the natural analogue of the one-shot problem, except that both the $v(\cdot)$ and $p(\cdot)$ functions account for the possible arrival of future financing-investment opportunities. The following proposition describes the equity and debt holders' problems with repeated investment

Proposition 6 (Repeated Investment). A solution consists of a value of equity $v(\ell)$, price $p(\ell)$, policies $\{g(\ell), m(\ell), \psi(\ell), \hat{\ell}(\ell)\}$, and default threshold, $\bar{\ell}$, such that

1. Given $v(\ell)$ and $p(\ell):(a)$ the policies $\{g(\ell), m(\ell), \psi(\ell), \hat{\ell}(\ell)\}$ solve the firm's problem in (20); (b) $v(\ell)$ satisfies the differential variational equation (DVI)

$$
\begin{equation*}
0=\min \left\{(r-\mu) v(\ell)+\mu \ell v^{\prime}(\ell)-\frac{\sigma^{2}}{2} \ell^{2} v^{\prime \prime}(\ell)-\lambda(v(\hat{\ell}(\ell))-v(\ell))-(1-r \ell), v(\ell)\right\} \tag{31}
\end{equation*}
$$

2. The default threshold $\bar{\ell}$ is optimal and is determined by the indifference in (31)
3. Given $v(\ell)$ and the equity holders' policies, the price $p(\ell)$ solves

$$
\begin{align*}
r p(\ell) & =r+\left(\sigma^{2}-\mu\right) \ell p^{\prime}(\ell)+\frac{\sigma^{2}}{2} \ell^{2} p^{\prime \prime}(\ell)+\lambda(p(\hat{\ell}(\ell))-p(\ell))  \tag{32}\\
p(\bar{\ell}) & =\frac{(1-\theta) v(0)}{\bar{\ell}} \tag{33}
\end{align*}
$$

Furthermore, the first-best investment choice as defined in Definition 1 is

$$
\begin{equation*}
g^{u}=\frac{1}{\zeta(r-\mu)\left(\frac{1}{2}\left(\sqrt{1-\frac{2 \lambda}{\zeta(r-\mu)^{2}}}-1\right)+1\right)} \tag{34}
\end{equation*}
$$

Proof. See Appendix C.
Unlike the one-shot case, we do not have closed-form solutions when $\lambda>0$. Thus, we need to solve the model numerically using upwind finite difference methods. To do so, we add artificial reflecting barriers to the stochastic process, $v^{\prime}\left(\ell_{\min }\right)=0, v^{\prime}\left(\ell_{\max }\right)=0$, and $p^{\prime}\left(\ell_{\min }\right)=0$. The absorbing boundary condition for $p(\cdot)$ comes from (33) (i.e., the liquidation value of the firm at the time of default).

### 4.3 Analysis

Figure 5 depicts investment relative to first-best ( $\tilde{g} \equiv g / g^{u}$ ) and post-investment leverage relative to its pre-investment level $(\hat{\ell} / \ell)$ plotted against leverage when $\lambda=0.2$ (left panel) and $\lambda=0.3$ (right panel), for different values of $\kappa$. It shows that the repeated arrival of financing-investment opportunities generates heterogeneous investment distortions, with low leverage firms tending to underinvest and high leverage firms tending to overinvest. ${ }^{21}$ This heterogeneous effect of limited liability on equity holders' investment decisions is our key finding. Figure 5 also shows that firms may overinvest even if they are deleveraging. This observation further highlights that our mechanism differs from the mechanisms emphasized in earlier work on risk-shifting and debt dilution. We next discuss the main intuition behind equity holders' investment and leverage choices

Equity holders' choice of investment Consider first the effect of an increase in the arrival rate of financing-investment opportunities, $\lambda$, on equity holders' investment decisions for a given level of $\kappa$. We see that a higher arrival of financing-investment opportunities decreases equity holders' investment relative to the first-best level for any level of $\kappa$. This is driven by two effects. First, a higher $\lambda$ implies that equity holders have more opportunities to dilute existing debt holders, if such dilution is profitable. Therefore, new creditors expect

[^15]

Figure 5: The effect of equity payout restrictions on the investment-leverage relationship. The top panels show investment relative to first-best ( $\tilde{g}$ ) against preexisting leverage $(\ell)$, while the bottom panels show the ratio of ex-post to ex-ante leverage $(\hat{\ell} / \ell)$. Each line corresponds to a different value for the constraint on equity payouts $\kappa$. The case $\kappa=0$ is the baseline case of no equity payouts from debt. The left and right panels show this for an arrival rate of new financing-investment opportunities $\lambda=0.2$ (left) and $\lambda=0.3$ (right). Bankruptcy costs are set to zero $(\theta=0)$. Vertical lines indicate the default thresholds for each value of $\kappa$. Parameter values are discussed in Appendix E.
their claims to be diluted sooner and require to be compensated for that, which increases the cost of debt financing. Second, an increase in $\lambda$ indirectly increases the cost of inefficient investment. ${ }^{22}$ The first of these forces is responsible for the large decrease in investment when $\kappa$ is high and among low leverage firms when $\kappa$ is low, while the second effect explains

[^16]a reduction in investment among high-leverage firms when $\kappa$ is low.
Consider next the effect of an increase in equity holders' ability to make equity payouts, $\kappa$, for a given level of $\lambda$. We see that an increase in $\kappa$ decreases the extent of equity holders' overinvestment. Moreover, this decrease is disproportionally larger for firms with low leverage inducing these firms to underinvest. This is because an increase in $\kappa$ decreases the cost of dilution as equity holders can now increase their leverage by increasing their equity payouts instead of engaging in inefficient investment. This effect is the strongest when $\ell$ is relatively low since low leverage firms have the largest capacity to increase leverage (and, hence, dilute debt holders). Since new creditors anticipate this behavior, the cost of debt increases sharply for low leverage firms', leading them to underinvest. As $\kappa$ increases, this mechanism becomes relevant also for firms with higher levels of leverage, and when $\kappa=1$ most firms underinvest.

Finally, note that for the highest levels of leverage, Figure 5 shows that investment converges to the first-best as $\kappa$ increases, similarly to the one-shot model. This is because for high enough $\ell$ and $\kappa$ equity holders are able to issue so much debt that they optimally choose to default immediately after investment. This implies that debt holders immediately take over the firm and equity holders have no opportunities to dilute new debt holders' claims. Thus, as long as there are no deadweight bankruptcy costs (or these costs are small), equity holders' and new debt holders' incentives are again aligned, just as in the model with one-shot investment.

Equity holders' choice of leverage Next, we consider equity holders' leverage policy given $\lambda$ and $\kappa$. We see that low leverage firms increase their leverage more aggressively than high-leverage firms. This is because low leverage firms find it easier to increase their leverage: their initial leverage is low so even a small debt issuance increases their leverage above its current level. On the other hand, the same amount of debt issuance for high leverage firms leads to a much more modest change in their post-investment leverage. Since increasing leverage is costly (because at least part of it is driven by inefficient investment whose costs are fully borne by equity holders), firms with high leverage adjust their leverage less aggressively than low leverage.

Somewhat more surprisingly, we see that when equity payout constraints are strict (i.e., when $\kappa$ is low), firms with high leverage choose to deleverage. This is driven by the fact that when inefficient investment is the main way to increase leverage the investment needed to increase leverage is an increasing function of $\ell .{ }^{23}$ Therefore, as $\ell$ increases, inefficient investment needed to increase leverage becomes exceedingly costly and, hence, firms with high $\ell$ choose to deleverage. However, note that this deleveraging is still associated with

[^17]overinvestment. This is because, on the margin, increasing investment above its efficient level and financing it with debt allows the equity holders to limit the value of cash flows generated by new investment that are captured by existing debt holders.

Finally, we see that an increase in $\kappa$ increases the post-investment leverage of all firms. An increase in $\lambda$ has a more subtle effect, but it tends to decrease the leverage of high leverage firms. This decrease is driven by the fact that the price of debt for high leverage firms actually increases when $\lambda$ increases since these firms deleverage when they invest, and investment is now more frequent. As such, when $\lambda$ increases these firms can finance their choices with less debt.

Comparison with one-shot investment Two general observations arise when comparing the above results with the results derived in the model with one-shot investment. First, the tendency of equity holders' to overinvest extends to the model with repeated financinginvestment opportunities, and the extent of overinvestment decreases with the arrival rate of new financing opportunities. Second, the model with repeated investment opportunities shows that allowing equity payouts financed with debt (i.e., $\kappa>0$ ) need not improve the efficiency of firms' investment and may even induce equity holders to switch from overinvestment to underinvestment. Intuitively, when $\kappa$ is high debt holders' concerns about the future dilution of their claims are exacerbated when equity holders can make direct payouts to themselves financed with debt more frequently.

Relation to existing literature It is useful to contrast our findings with predictions based on other models of financial frictions and with existing literature on debt overhang. Models of financial frictions that feature collateral constraints also assume that firms are protected by limited liability. However, these models also feature additional sources of market incompleteness such as private information (e.g., Bernanke et al. (1999) or Clementi and Hopenhayn (2006)) or ability to abscond funds (e.g., Buera et al. (2011) or Moll (2014)). In those models, the ability to issue equity payouts would have no effect on constrained firms' investment choices since, in those models, it is typically optimal to delay dividends (see, for example, Clementi and Hopenhayn (2006)). In addition, a higher arrival of financinginvestment opportunities would either have no effect (if collateral constraints are modeled as in Buera et al. (2011) or Moll (2014)) or would lead to an increase in investment relative to first-best due to an implied increase in firms' future profitability (if collateral constraints are modeled as in Clementi and Hopenhayn (2006)). In contrast, we find that equity payouts tend to decrease firms' investment relative to first-best, resulting in underinvestment by low leverage firms but limiting overinvestment by high-leverage firms. Furthermore, in our model, a higher arrival rate unambiguously decreases investment relative to the first-best
for all firms.
Our results are also related to the literature on debt overhang in dynamic models based on Leland (1994). This literature emphasized that in the presence of existing debt firms' underinvest, as predicted by Myers (1977) (see, for example, Diamond and He (2014)). These papers assume that equity holders finance their investment with equity. In contrast, we allow equity holders to choose their financing optimally. Our predictions for real investment also differ from DeMarzo and He (2021), who find that equity holders underinvest even though they can finance their investment with debt. In their investigation of real investment DeMarzo and He (2021) assume that equity holders can continuously adjust their leverage and that bankruptcy results in a complete loss of the firm's value. With zero recovery value, the "double selling" mechanism is absent-and so equity holders cannot directly benefit from indirect dilution due to changes in the default timing. Indeed, in our model if bankruptcy losses are very high, equity holders would also find it optimal to always underinvest in our model.

Covenants In the model, for simplicity, we assumed that existing debt is not protected by covenants. In Appendix D, we show that standard covenants such as restrictions on payouts, secure debt restrictions, restrictions on leverage, and senior debt restrictions do not resolve the time-inconsistency issue we highlight in our paper.

## 5 Conclusion

This paper provides a new dynamic model showing that limited liability can inefficiently distort investment away from low leverage firms and towards highly levered firms when access to financial markets is not frictionless. Taken together, in our model high leverage firms have an incentive to overinvest because debt financed investment allows them to (1) dilute current debt holders' coupon claims by increasing leverage and bringing forward bankruptcy; or (2) limit the gains from investment to debt holders from new investment if increasing leverage is too costly. At the same time, our model predicts that low leverage firms with frequent financing opportunities tend to underinvest. This is because these firms have the largest capacity to dilute the coupons of debt holders in the future, which is anticipated by creditors who require high compensation for lending to those firms. These mechanisms are robust if all debt is collateralized and strengthen when equity holders can deplete equity by making equity payouts from new debt issuances.

The analysis in this paper has important policy implications - especially during times with enormous government support for corporations, as in the response to the Covid-19 crisis and the financial crisis of 2008-2009. Our model emphasizes that government programs that
increase firms' debt burden may have undesired consequences, as higher leverage induces debt financed overinvestment among highly levered firms. Another lesson from our analysis is that restrictions on equity payouts are no cure-all to excessive leverage and low investment, and the effects of such a policy are heterogenous by initial firm leverage. In our model, equity payout restrictions reduce bankruptcy and may raise investment towards the firstbest when initial leverage is low. However, equity payout restrictions tend to exacerbate inefficient overinvestment when initial firm leverage is high.

To emphasize the role of limited liability, we abstracted from other related mechanism and focused on decisions of a single firm. The model can be extended to allow for information frictions in the quality of collateral (Gorton and Ordoñez (2014)). If collateral quality is cyclical, debt financing would likely be further distorted relative to our benchmark. The channel emphasized in this paper likely has potentially important aggregate implications, both through investment distortions across firms (e.g., Khan and Thomas (2013), Moll (2014), and Buera et al. (2011)), and the cyclicality of the costs of financial distress (e.g. Atkeson et al. (2017)). We believe that investigating these general equilibrium consequences of limited liability will be fruitful.

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## Appendix A Proofs for Section 2

## A. 1 Proof of Proposition 1

Proof of Proposition 1. To find the post-investment value of equity $V(Z, L)$ that solves equity holders' default problem as described by (5)- (7) we use the method of undetermined coefficients with the guess

$$
\begin{equation*}
V(Z, L)=\frac{1}{r-\mu}\left(Z+\frac{\omega}{\eta} Z^{-\eta}\right)-L \tag{A.1}
\end{equation*}
$$

Using this guess in (5) and equating undetermined coefficients we arrive at the equation $2(r+\eta \mu)=$ $\eta(1+\eta) \sigma^{2}$. We solve this quadratic equation for $\eta$ and note that the smaller of the two roots is explosive and, hence, it violates the transversality condition. Therefore,

$$
\begin{equation*}
\eta=\frac{\left(\mu-\sigma^{2} / 2\right)+\sqrt{\left(\mu-\sigma^{2} / 2\right)^{2}+2 \sigma^{2} r}}{\sigma^{2}} \tag{A.2}
\end{equation*}
$$

Next, we substitute the guess (A.1) into (7) to find that $\omega=\underline{Z}^{\eta+1}$. Then we use (A.1) and the expression for $\omega$ in (6) to find that the default threshold is given by

$$
\begin{equation*}
\underline{Z}=\frac{(r-\mu) \eta}{\eta+1} L \tag{A.3}
\end{equation*}
$$

(A.3) defines the default threshold and completes derivations of $V(Z, L)$. Since the default threshold depends on equity holders' liabilities $L$ we donate it by $\underline{Z}(L)$.

Next, we show that the value of equity can be expressed as $V(Z, L)=v(\ell) Z$, where $\ell \equiv L / Z$, and derive the expression for $v(\cdot)$. First, we note that

$$
\begin{equation*}
\frac{\underline{Z}(L)}{Z}=\frac{(r-\mu) \eta}{\eta+1} \frac{L}{Z}=\frac{(r-\mu) \eta}{\eta+1} \ell \tag{A.4}
\end{equation*}
$$

Next, we use the expressions for $\omega$ and for $\underline{Z} / Z$ found in (A.1) to obtain

$$
\begin{equation*}
V(Z, L) / Z=\frac{1}{r-\mu}+\frac{\chi}{\eta+1} \ell^{\eta+1}-\ell \tag{A.5}
\end{equation*}
$$

where

$$
\begin{equation*}
\chi \equiv\left(\frac{(r-\mu) \eta}{\eta+1}\right)^{\eta} \tag{A.6}
\end{equation*}
$$

Finally, we set $v(\ell)=\frac{1}{r-\mu}-\ell(1-s(\ell))$, were $s(\ell)=\frac{\chi}{\eta+1} \ell^{\eta}$, which implies that $V(Z, L) / Z=v(\ell)$.
To find the liquidation value of the firm we note that equity holders walk away when cash flows $Z$ reach the default threshold $\underline{Z}(L)$. At that time debt holders take over the firm so its liabilities are reset to $L=0$ but the firm loses fraction $\theta \in[0,1]$ of its value. It follows that the liquidation
value of the firm, from creditors' perspective, is given by

$$
\begin{equation*}
V((1-\theta) \underline{Z}(L), 0)=\frac{(1-\theta) \underline{Z}(L)}{r-\mu} \tag{A.7}
\end{equation*}
$$

Thus, the liquidation value per unit of liabilities is given by

$$
\begin{equation*}
\frac{V((1-\theta) \underline{Z}(L), 0)}{L}=\frac{(1-\theta) \eta}{\eta+1} \tag{A.8}
\end{equation*}
$$

where we used the definition of $\underline{Z}(L)$ (see (A.3).

## A. 2 Proof of Proposition 2

Proof of Proposition 2. (Derivations of debt prices) Let $T$ be the first-time cash flows, $Z$, reach the default threshold $\underline{Z}(L)$. Since $Z$ follows a geometric Brownian motion (see (1)) we have

$$
\begin{equation*}
\mathbb{E}_{T}\left[e^{-r T}\right]=\exp \left(\frac{-\left(\mu-\sigma^{2} / 2\right)-\sqrt{\left(\mu-\sigma^{2} / 2\right)^{2}}+2 \sigma^{2} r}{\sigma^{2}}(\log Z-\log \underline{Z}(L))\right) \tag{A.9}
\end{equation*}
$$

as shown in Jeanblanc et al. (2009). Using the definition of $\eta$ and $\chi$ (see (A.2) and (A.6), respectively) and the expression obtained in (A.3) we conclude that

$$
\begin{equation*}
\mathbb{E}_{T}\left[e^{-r T}\right]=(Z / \underline{Z}(L))^{-\eta}=\chi \ell^{\eta} \tag{A.10}
\end{equation*}
$$

Using (A.10) we see that

$$
\begin{equation*}
P^{C}(Z, L) \equiv \frac{p^{C}(Z, L)}{r}=\frac{1}{r}\left[1-\chi \ell^{\eta}\right]=\frac{1}{r}[1-(1+\eta) s(\ell)] \tag{A.11}
\end{equation*}
$$

where $s(\ell)=\chi /(1+\eta) \ell^{\eta}$ and $\ell=L / Z$. Note that the above equation implies that the relevant state variable is $\ell$. Hence, the price of coupon claims depends only in $\ell$, and we can denote that price by $P^{C}(\ell)$.

Next, we consider the price of a bankruptcy claim $P^{B}(Z, L)$. Note that

$$
\begin{equation*}
P^{B}(Z, L) \equiv \frac{p^{B}(Z, L)}{r}=\frac{V((1-\theta) \underline{Z}(L), 0)}{r L} \mathbb{E}_{T}\left[e^{-r T}\right]=\frac{1}{r}(1-\theta) \eta s(\ell) \tag{A.12}
\end{equation*}
$$

where we used (A.8), (A.9), and the definition of $s(\ell)$. We see again that the relevant state variable is $\ell$ and, thus, we can write the price of claims bankruptcy as $P^{B}(\ell) \equiv \frac{p^{B}(\ell)}{r}$.

From the above discussion it follows that the leverage $\ell$ is the relevant state for pricing defaultable consols so that we can write $P(Z, L)=P(\ell)$ and $p(Z, L)=p(\ell)$. Putting together (A.11) and (A.12) we obtain

$$
\begin{equation*}
P(\ell)=\frac{p(\ell)}{r}=\frac{1}{r}[1-(1-\theta \eta) s(\ell)] \tag{A.13}
\end{equation*}
$$

(The Budget Constraint) Equity holders issue debt to finance their equity payouts, $M$, and a fraction $\psi$ of the investment cost $Z q(g)$. Let $K$ denote the quantity of new bonds issued by equity holders to finance $\psi Z q(g)+M$. Then, $K$ has to satisfy the following budget constraint

$$
\begin{equation*}
P(\hat{\ell}) K=\psi Z q(g)+M \tag{A.14}
\end{equation*}
$$

where $\hat{\ell}$ is the post-investment leverage. Next, we relate $K$ to the change in leverage $\hat{L}-L$. Recall that $L$ is defined as the present discounted value (PDV) of liabilities. Since each unit of debt promises a payment of a constant coupon of 1 and agents discount these payments at a rate $r$ it follows the PDV of the cash flows promised to new debt holders is given by $K / r$. Therefore, the post-investment liabilities are given by $\hat{L}=L+\frac{K}{r}$. It follows that $K=r(\hat{L}-L)$. Substituting this expression for $K$ into the budget constraint (A.14), dividing both sides of the resulting equation by $Z$, and using the definition of $p(\cdot)$ (see (A.13)) we obtain

$$
\begin{equation*}
p(\hat{\ell})(\hat{\ell}(1+g)-\ell)=\psi q(g)+m \tag{A.15}
\end{equation*}
$$

where $\hat{\ell}=\hat{L} /(Z(1+g))$. This corresponds to (18) in the text.
(Collateralized Debt) We formally show that (i) the price of debt where default claims are pledged as collateral (as opposed to the proportional claims to the firm in bankruptcy) is identical to the price of debt in our baseline model under suitable pledgeability constraint and (ii) under pledgeability constraints, equity holders choices respect constraint (19) that we impose in the benchmark model.

To consider collateralized debt we interpret the value of the firm in bankruptcy, $\frac{\eta}{1+\eta} L$, as pledgeable collateral. ${ }^{24,25}$ Let $C$ denote the collateral promised to the existing debt holders. We assume that the firm used all of its assets in available in bankruptcy as collateral when it issued existing debt in the past -as would be optimal - so that $C=\frac{\eta}{1+\eta} L$. In order for the equity holder to deliver this value following their investment and financing choices, it has to be the case that

$$
\begin{equation*}
\frac{\eta}{1+\eta} \hat{L} \geq \frac{\eta}{1+\eta} L \tag{A.16}
\end{equation*}
$$

and that only

$$
\begin{equation*}
\min \left\{\frac{\hat{Z}}{r-\mu}, \frac{\eta}{1+\eta} \hat{L}\right\}-\frac{\eta}{1+\eta} L \tag{A.17}
\end{equation*}
$$

can be used for collateral for the new debt. ${ }^{26}$ We refer to (A.16) and (A.17) as the pledgeability

[^18]constraints.
Now, consider the price of new collateralized debt, denoted by $\hat{P}^{S}(\hat{Z}, \hat{L}, \hat{C})$, where $\hat{Z}$ and $\hat{L}$ are post-investment cash flows and liabilities, respectively, and $\hat{C}$ is the value of collateral promised to new debt holders. Then, following the same argument as we used in the proof of Proposition 2, we obtain
\[

$$
\begin{equation*}
\hat{P}^{S}(\hat{Z}, \hat{L}, \hat{C})=P^{C}(L, Z)+\frac{\hat{C}}{r(\hat{L}-L)} \chi\left(\frac{\hat{L}}{\hat{Z}}\right)^{\eta}, \tag{A.18}
\end{equation*}
$$

\]

where $P^{C}(L, Z)$ is defined in (14). Since the value of new debt is increasing in collateral promised and equity holders do not get to keep any collateral unused, it follows that $\hat{C}=\min \left\{\frac{\hat{Z}}{r-\mu}, \frac{\eta}{1+\eta} \hat{L}\right\}-\frac{\eta}{1+\eta} L$. Therefore, if $\frac{\eta}{1+\eta} \hat{L} \leq \frac{\hat{Z}}{r-\mu}$ (as we argue below), then

$$
\begin{equation*}
\hat{P}^{S}(\hat{Z}, \hat{L}, \hat{C})=P^{C}(\hat{L}, \hat{Z})+\frac{1}{r} \frac{\eta}{1+\eta} \chi\left(\frac{\hat{L}}{\hat{Z}}\right)^{\eta}=P^{C}(\hat{L}, \hat{Z})+P^{B}(\hat{L}, \hat{Z})=P(\hat{L}, \hat{Z}) \tag{A.19}
\end{equation*}
$$

This implies that the price of new debt is exactly the same as in the benchmark model. A similar argument can be used to show that the price of existing collateralized debt, which we denote by $P^{S}(Z, L, C)$, is equal to $P(Z, L)$.

We now argue that equity holders find it optimal to choose $\hat{L} \leq \frac{\hat{Z}}{r-\mu}$. To see that this is the case, note that choosing $\hat{L}>\frac{\hat{Z}}{r-\mu}$ yields the same payoff as $\hat{L}=\frac{\hat{Z}}{r-\mu}$. This is because in both cases equity holders decide to walk away from the firm and the total value of the new debt issued is the same. From the above discussion, we know that if $\hat{L} \leq \frac{\hat{\mathcal{Z}}}{r-\mu}$ then equity holders' problem is the same as in the baseline model. Thus, if $\kappa=0$ or $\kappa<\underline{\kappa}$ (where $\underline{\kappa}$ is defined as in Proposition 5) then equity holders strictly prefer to choose $\hat{L} \leq \frac{\hat{Z}}{r-\mu}$. If $\kappa \geq \underline{\kappa}$ then equity holders are indifferent between any $\hat{L} \geq \frac{\hat{Z}}{r-\mu}$ and so we can assume that they choose $\hat{L}=\frac{\hat{Z}}{r-\mu}$. This completes our argument.

## A. 3 Proof of Proposition 3

Proof of Proposition 3. To derive (24) consider equity holders' objective function (20) and substitute the budget constraint ((A.15)) to eliminate $\psi q(g)+m$ and obtain

$$
\begin{equation*}
(1+g) v(\hat{\ell})-q(g)+p(\hat{\ell})(\hat{\ell}(1+g)-\ell) \tag{A.20}
\end{equation*}
$$

Using the expression we found for $v(\hat{\ell})$ (Proposition 1), (A.20), the observation $p(\hat{\ell})=1-(1+\theta \eta) s(\hat{\ell})$, and simplifying we obtain

$$
\begin{equation*}
\frac{1+g}{r-\mu}-p(\hat{\ell}) \ell-q(g)-\theta \eta s(\hat{\ell}) \hat{\ell}(1+g) \tag{A.21}
\end{equation*}
$$

value of the firm (if they choose to default immediately), $\frac{\hat{Z}}{r-\mu}$, net of the collateral promised to existing debt holders. This happens if equity holders choose $\frac{\eta}{1+\eta} \hat{L}>\frac{\hat{Z}}{r-\mu}$. Otherwise, if equity holders choose optimally to continue operating the firm, the maximal value of collateral they can promised to new debt holders is $\frac{\eta}{1+\eta}(\hat{L}-L)$.

Defining $H(\hat{\ell})=\theta \eta s(\hat{\ell})$ and using this definition in (A.21) we obtain the equity holders' objective function (24) in Proposition 3.

To obtain (27), note that $v(0)=\frac{1}{r-\mu}$. Thus, (23) implies that $g^{u}$ is a unique solution to the F.O.C. given by $0=\frac{1}{r-\mu}-q^{\prime}\left(g^{u}\right)$.

## Appendix B Proofs for Section 3

In this section, we provide proofs of propositions stated in Section 3 (Propositions 4 and 5). We begin by establishing a number of useful preliminary results and by discussing feasibility of equity holders' choice of $\{g, \hat{\ell}, m, \psi\}$ (Appendix B.1).

## B. 1 Preliminary Results

We establish first a number of preliminary results that we will use to prove Propositions 4 and 5 . Readers primarily interested in main results may choose to skip this section.

Lemma 1. Define $\bar{\ell}$ as the unique solution to

$$
\begin{equation*}
1=\chi \bar{\ell}^{\eta} \tag{B.1}
\end{equation*}
$$

Then $p(\bar{\ell}) \bar{\ell}=\frac{1}{r-\mu}$.
Proof. Plugging $\bar{\ell}$ into the expression for $p(\ell)$ (see (15)) we obtain $p(\bar{\ell})=\frac{\eta}{1+\eta}$. From the definitions of $\bar{\ell}$ and $\chi$ ((B.1) and (9), respectively) we obtain

$$
\begin{equation*}
\bar{\ell}=\frac{1+\eta}{\eta} \frac{1}{r-\mu} \tag{B.2}
\end{equation*}
$$

Combining these observations we obtain $p(\bar{\ell}) \bar{\ell}=\frac{1}{r-\mu}$.
In the proof of Lemma 1 we derived expressions for $\bar{\ell}$ and $p(\bar{\ell})$. Since we make use of these expressions repeatedly, we report them in the following Corollary.

Corollary 2. We have $\bar{\ell}=\frac{1}{r-\mu} \frac{1+\eta}{\eta}$ and $p(\bar{\ell})=\frac{\eta}{1+\eta}$
Next, we show the constraint $p(\ell) \geq p^{B}(\ell)$ is satisfied if and only if $\ell \in[0, \bar{\ell}]$.
Corollary 3. We have $p(\ell) \geq p^{B}(\ell)$ if and only if $\ell \in[0, \bar{\ell}]$.
Proof. From the definitions of $p(\ell)$ and $p^{B}(\ell)$ (see Proposition 2) we have $p(\ell)-p^{B}(\ell)=1-\chi \ell^{\eta}$. Since $\chi>0$ and $\eta>0$ it follows that $p(\ell)-p^{B}(\ell)$ is strictly decreasing in $\ell$. Moreover, from the definition of $\bar{\ell}$ we see that $p(\bar{\ell})-p^{B}(\bar{\ell})=0$. This establishes the claim.

Having characterized the highest feasible leverage, $\bar{\ell}$, we now establish two useful results regarding the value of outstanding debt.

Lemma 4. For all $\hat{\ell}<\bar{\ell}$ we have

$$
\begin{equation*}
\frac{\partial}{\partial \hat{\ell}}[p(\hat{\ell})(\hat{\ell}(1+g)-\ell)]=p^{\prime}(\hat{\ell})(\hat{\ell}(1+g)-\ell)+p(\hat{\ell})(1+g)>0 \tag{B.3}
\end{equation*}
$$

Proof. We have

$$
\begin{equation*}
\frac{\partial}{\partial \hat{\ell}}[p(\hat{\ell})(\hat{\ell}(1+g)-\ell)]=\left[1-\chi \hat{\ell}^{\eta}\right](1+g)+\frac{\eta \chi}{1+\eta} \hat{\ell}^{\eta-1} \ell \tag{B.4}
\end{equation*}
$$

where we used the definition of $p(\ell)$ (see (15) with $\theta=0$ ). The claim follows from the observation that, since $\hat{\ell}<\bar{\ell}$, we have $\left[1-\chi \hat{\ell}^{\eta}\right]>0$.

Lemma 5. The value of outstanding debt, $p(\ell) \ell$, is strictly increasing in $\ell$ for all $\ell \in[0, \bar{\ell})$.
Proof. We have

$$
\begin{equation*}
\frac{\partial}{\partial \ell} p(\ell) \ell=p^{\prime}(\ell) \ell+p(\ell)=1-\chi \ell^{\eta}>0 \tag{B.5}
\end{equation*}
$$

where the last inequality follows since $\ell<\bar{\ell}$.
Finally, we discuss which equity holders' choices of $g, \psi, m$, and $\hat{\ell}$ are feasible (i.e., satisfy equity holders' budget constraint). Note that the equity holders' choices of $g, \psi, m$, and $\hat{\ell}$ have to jointly satisfy the equity holders' budget constraint

$$
\begin{equation*}
p(\hat{\ell})((1+g) \hat{\ell}-\ell)=\psi q(g)+m \tag{B.6}
\end{equation*}
$$

The budget constraint implies that once the equity holders make choices of $g, \psi$, and $m$ the postinvestment leverage $\hat{\ell}$ is determined implicitly by (B.6). Thus, we can treat the post-investment leverage as an implicit function of $g, \psi$, and $m$, and denote it by $\hat{\ell}(g, \psi, m)$. This leads to the following definition of feasibility of equity holders choices.

Definition 2. The equity holders choices of $g, \psi, m$ are feasible if $\hat{\ell}(g, \psi, m) \leq \bar{\ell}$
We now derive a feasibility constraint on issuance of $m$.
Lemma 6. Fix $g$ and $\psi$ such that $\hat{\ell}(g, \psi, 0)<\bar{\ell}$. Then $m$ is feasible if $m \in[0, \bar{m}(g, \psi)]$, where

$$
\begin{equation*}
\bar{m}(g, \psi)=\frac{1+g}{r-\mu}-\frac{\eta}{1+\eta} \ell-\psi q(g) \tag{B.7}
\end{equation*}
$$

Moreover, at $m=\bar{m}(g, \psi)$ we have $\hat{\ell}(g, \psi, m)=\bar{\ell}$.
Proof. Since $\psi$ and $g$ are such that $\hat{\ell}(g, \psi, 0)<\bar{\ell}$ then, given choices of $g$ and $\psi$ there exists $m>0$ that satisfies

$$
\begin{equation*}
p(\hat{\ell})(\hat{\ell}(1+g)-\ell)=\psi q(g)+m \tag{B.8}
\end{equation*}
$$

By applying the implicit function theorem to the above equation we see that

$$
\begin{equation*}
\frac{\partial \hat{\ell}}{\partial m}=\frac{1}{p^{\prime}(\hat{\ell})(\hat{\ell}(1+g)-\ell)+p(\hat{\ell})(1+g)}>0 \tag{B.9}
\end{equation*}
$$

where the inequality follows from Lemma 4. It follows that at the highest feasible $m$, which we denote by $\bar{m}(g, \psi)$, we have $\hat{\ell}=\bar{\ell}$. Setting $\hat{\ell}=\bar{\ell}$ in (B.8) and rearranging, we obtain

$$
\bar{m}(g, \psi)=\frac{1+g}{r-\mu}-\frac{\eta}{1+\eta} \ell-\psi q(g)
$$

## B. 2 Proof of Proposition $4(\kappa=0)$

Proof of Proposition 4. (Part 1) Equity financing implies that $\psi=0$. Therefore, the post-investment leverage $\hat{\ell}$ is given by $\hat{\ell}=\frac{\ell}{1+g}$ and equity holders' problem simplifies to

$$
\max _{g \geq 0} \frac{1+g}{r-\mu}-p\left(\frac{\ell}{1+g}\right) \ell-q(g)
$$

The first-order condition associated with the above problem is given by

$$
\begin{equation*}
\frac{1}{r-\mu}+p^{\prime}\left(\frac{\ell}{1+g}\right)\left(\frac{\ell}{1+g}\right)^{2}-q^{\prime}(g)=0 \tag{B.10}
\end{equation*}
$$

Recall that $g^{u}$ denotes the first-best investment (see Definition 1) and let $g_{e}^{*}$ denote equity holders' optimal investment under equity financing. Then, (27) and (B.10) imply that

$$
\begin{equation*}
\frac{1}{r-\mu}-q^{\prime}\left(g^{u}\right)=0=\frac{1}{r-\mu}+p^{\prime}\left(\frac{\ell}{1+g_{e}^{*}}\right)\left(\frac{\ell}{1+g_{e}^{*}}\right)^{2}-q^{\prime}\left(g_{e}^{*}\right)<\frac{1}{r-\mu}-q^{\prime}\left(g_{e}^{*}\right) \tag{B.11}
\end{equation*}
$$

where the inequality follows from the observation that $p^{\prime}(\ell)<0$ for all $\ell$. Since the cost function $q$ is strictly increasing in $g$, (B.11) implies that $g_{e}^{*}<g^{u}$.
(Part 2) We now allow the equity holders to choose their financing of investment optimally so that $\psi \in[0,1]$. In this case, the equity holders' problem is given by

$$
\begin{align*}
\max _{\substack{g, \hat{\ell} \geq 0 \\
\psi \in[0,1]}} & \frac{1+g}{r-\mu}-p(\hat{\ell}) \ell-q(g)  \tag{B.12}\\
\text { s.t. } & p(\hat{\ell})(\hat{\ell}(1+g)-\ell)=\psi q(g)  \tag{B.13}\\
& p(\hat{\ell}) \geq p^{B}(\hat{\ell}) \tag{B.14}
\end{align*}
$$

where $m=0$ since $\kappa=0$. Recall that $\hat{\ell}(g, \psi, m)$ denotes the level of post-investment leverage implied by equity holders' choices via the budget constraint. Since $m=0$ in what follows we slightly abuse notation and write $\hat{\ell}(g, \psi)$ instead to $\hat{\ell}(g, \psi, 0)$.

It is easy to see that, as long as $\ell<\bar{\ell}$ the equity holders will never choose $g, \psi$ such that $\hat{\ell}(g, \psi)=\bar{\ell}$. This is because when $\hat{\ell}(g, \psi)=\bar{\ell}$ then equity holders' post-investment value of equity is 0 (equity holders immediately default) while $\ell<\bar{\ell}$ implies that the pre-investment value of equity is strictly positive. It follows that the constraint (B.14) (which, as shown in Corollary 3 is equivalent to the constraint $\hat{\ell} \leq \bar{\ell}$ ) is not binding. Hence, the equity holders' problem can be written as

$$
\begin{equation*}
\max _{\substack{g \geq 0 \\ \psi \in[0,1]}} \frac{1+g}{r-\mu}-p(\hat{\ell}(g, \psi)) \ell-q(g), \tag{B.15}
\end{equation*}
$$

subject to $\psi \in[0,1]$, where $\hat{\ell}(g, \psi)$ is implicitly defined by (B.13). Note that the first-order derivative of equity holders' objective function (B.15) w.r.t. $\psi$ is given by

$$
\begin{equation*}
-p^{\prime}(\hat{\ell}) \ell \frac{\partial \hat{\ell}}{\partial \psi}>0 \tag{B.16}
\end{equation*}
$$

since $p^{\prime}(\hat{\ell})<0$ and $\partial \hat{\ell} / \partial \psi$ (obtained by applying the implicit function theorem to (B.13)) is given by

$$
\begin{equation*}
\frac{\partial \hat{\ell}}{\partial \psi}=\frac{q(g)}{p^{\prime}(\hat{\ell})(\hat{\ell}(1+g)-\ell)+p(\hat{\ell})(1+g)}>0 \tag{B.17}
\end{equation*}
$$

It follows that equity holders find it optimal to finance all of their investment with debt, that is, $\psi^{*}=1$.

Consider next the optimal choice of $g$. The optimal choice of $g$, which we denote by $g^{*}$, satisfies the following first-order condition

$$
\begin{equation*}
\frac{1}{r-\mu}-\left.p^{\prime}(\hat{\ell}) \ell \frac{\partial \hat{\ell}}{\partial g}\right|_{\substack{\psi=1 \\ g=g^{*}}}-q^{\prime}\left(g^{*}\right)=0 \tag{B.18}
\end{equation*}
$$

where

$$
\begin{equation*}
\frac{\partial \hat{\ell}}{\partial g}=-\frac{p(\hat{\ell}) \hat{\ell}-\psi q^{\prime}(g)}{p^{\prime}(\hat{\ell})(\hat{\ell}(1+g)-\ell)+p(\hat{\ell})(1+g)} \tag{B.19}
\end{equation*}
$$

We now argue that $\partial \hat{\ell} / \partial g$ evaluated at $g=g^{*}, \psi=\psi^{*}=1$ is strictly positive. To see this, note that (B.18) implies that

$$
-q^{\prime}\left(g^{*}\right)=-\frac{1}{r-\mu}+\left.p^{\prime}(\hat{\ell}) \ell \frac{\partial \hat{\ell}}{\partial g}\right|_{\substack{\psi=1 \\ g=g^{*}}}
$$

Using the above expression in (B.19) evaluated at $g=g^{*}, \psi=1$ and rearranging, we obtain

$$
\begin{equation*}
\left.\frac{\partial \hat{\ell}}{\partial g}\right|_{\substack{\psi=1 \\ g=g^{*}}}\left[\frac{\left(1+g^{*}\right)\left(p(\hat{\ell})+p^{\prime}(\hat{\ell}) \hat{\ell}\right)}{p^{\prime}(\hat{\ell})\left(\hat{\ell}\left(1+g^{*}\right)-\ell\right)+p(\hat{\ell})\left(1+g^{*}\right)}\right]=\frac{-p(\hat{\ell}) \hat{\ell}+\frac{1}{r-\mu}}{p^{\prime}(\hat{\ell})\left(\hat{\ell}\left(1+g^{*}\right)-\ell\right)+p(\hat{\ell})\left(1+g^{*}\right)} \tag{B.20}
\end{equation*}
$$

From Lemma 1 and Lemma 5 we know that $-p(\hat{\ell}) \hat{\ell}+\frac{1}{r-\mu} \geq 0$ with a strict inequality if $\hat{\ell}<\bar{\ell}$. However, as we argued above, choosing $\hat{\ell}=\bar{\ell}$ is not optimal. Thus, at optimal choices of investment and financing we have $-p(\hat{\ell}) \hat{\ell}+\frac{1}{r-\mu}>0$. Next, note that by Lemma 4 the denominator on the RHS of (B.20) is strictly positive. Thus, it follows that the RHS of (B.20) is strictly positive. Furthermore, (B.20). Lemmas 4 and 5 imply that the expression in square brackets on the LHS of (B.20) is strictly positive. Therefore, we conclude that

$$
\begin{equation*}
\left.\frac{\partial \hat{\ell}}{\partial g}\right|_{\substack{\psi=g^{*}}}>0 \tag{B.21}
\end{equation*}
$$

Having established that $\partial \hat{\ell} /\left.\partial g\right|_{\left\{g=g^{*}, \psi=1\right\}}>0$ we consider again (B.18). Since, $p^{\prime}(\hat{\ell})<0$ we have

$$
\begin{equation*}
0=\frac{1}{r-\mu}-q^{\prime}\left(g^{*}\right)-\left.p^{\prime}(\hat{\ell}) \ell \frac{\partial \hat{\ell}}{\partial g}\right|_{\substack{\psi=1 \\ g=g^{*}}}>\frac{1}{r-\mu}-q^{\prime}\left(g^{*}\right) \tag{B.22}
\end{equation*}
$$

Since the cost function $q$ is strictly increasing, (B.22) implies that $g^{*}>g^{u}$.

## B. 3 Proof of Proposition $5(\kappa>0)$

Before we prove Proposition 5, we establish an important intermediate result.
Lemma 7. Equity holders choose to issue as much dividend as they can. That is, given $g$ and $\psi$ such that $\hat{\ell}(g, \psi, 0)<\bar{\ell}$, the equity holders choose

$$
\begin{equation*}
m^{*}=\min \{\kappa, \bar{m}(g, \psi)\} \tag{B.23}
\end{equation*}
$$

where $\bar{m}(g, \psi)$ is defined in (B.7).
Proof. The first-derivative of equity holders' objective function (B.12) w.r.t. $m$ is given by

$$
-p^{\prime}(\hat{\ell}) \ell \frac{\partial \hat{\ell}}{\partial m}>0
$$

since $p^{\prime}(\hat{\ell})<0$ and $\partial \hat{\ell} / \partial m>0$ (see the proof of Lemma 6 ). Thus, it follows that $m^{*}=\min \{\kappa, \bar{m}(g, \psi)\}$

Lemma 7 tells us that equity holders' problem can be simplified to

$$
\begin{align*}
\max _{\substack{g \geq 0 \\
\psi \in[0,1]}} & \frac{1+g}{r-\mu}-p(\hat{\ell}) \ell-q(g)  \tag{B.24}\\
\text { s.t. } & p(\hat{\ell})(\hat{\ell}(1+g)-\ell)=\psi q(g)+m^{*}  \tag{B.25}\\
& p(\hat{\ell}) \geq p^{B}(\hat{\ell})  \tag{B.26}\\
& m^{*}=\min \{\kappa, \bar{m}(g, \psi)\} \tag{B.27}
\end{align*}
$$

In other words, we can think of equity holders' problem as choosing first $g$ and $\psi$ and then setting $m$ to the highest value that is feasible.

Proof of Proposition 5. The proof consists of four parts. First, we characterize the solution when $\kappa=\infty$. We refer to this solution as the "unconstrained" solution. Next, we determine $\bar{\kappa}$ such that if $\kappa \geq \bar{\kappa}$ then the unconstrained solution is attainable. Thus, for all $\kappa \geq \bar{\kappa}$ the equity payout constraint, $m \leq \kappa$, is not binding. We then show that there exists $\underline{\kappa}$ with $\underline{\kappa}<\bar{\kappa}$ such that if $\kappa \in[\underline{\kappa}, \bar{\kappa}]$ then the equity holders can still attain the same payoff as in the case of $\kappa=\infty$ but with an additional restriction on their financing choices. Finally, we determine the equity holders' choices when $\kappa<\underline{\kappa}$.

When $\kappa=\infty$ then $m^{*}=\bar{m}(g, \psi)$ (see Lemma 7). Then the first-order derivative of equity holders' objective function w.r.t. $\psi$ is given by

$$
\begin{equation*}
-p^{\prime}(\hat{\ell})\left[\frac{\partial \hat{\ell}}{\partial \psi}+\frac{\partial \hat{\ell}}{\partial m} \frac{\partial m^{*}}{\partial \psi}\right] \tag{B.28}
\end{equation*}
$$

Since $m^{*}=\bar{m}(g, \psi)$, we have that

$$
\begin{equation*}
\frac{\partial \bar{m}(\psi, g)}{\partial \psi}=-q(g) \tag{B.29}
\end{equation*}
$$

Moreover,

$$
\begin{align*}
\frac{\partial \hat{\ell}}{\partial \psi} & =\frac{q(g)}{p(\hat{\ell})(1+g)+p^{\prime}(\hat{\ell})(\hat{\ell}(1+g)-\ell)}  \tag{B.30}\\
\frac{\partial \hat{\ell}}{\partial m} & =\frac{1}{p(\hat{\ell})(1+g)+p^{\prime}(\hat{\ell})(\hat{\ell}(1+g)-\ell)} \tag{B.31}
\end{align*}
$$

Therefore, it follows that

$$
\begin{equation*}
p^{\prime}(\hat{\ell})\left[\frac{\partial \hat{\ell}}{\partial \psi}+\frac{\partial \hat{\ell}}{\partial m} \frac{\partial m^{*}}{\partial \psi}\right]=0 \tag{B.32}
\end{equation*}
$$

That is, when equity holders' equity payout choices are unconstrained they are indifferent between any $\psi \in[0,1]$.

Fix $\psi^{*} \in[0,1]$. Consider now the first-order condition that determines the optimal investment, $g^{*}$, and which is given by

$$
\begin{equation*}
\frac{1}{r-\mu}+p^{\prime}(\hat{\ell}) \ell\left[\left.\frac{\partial \hat{\ell}}{\partial g}\right|_{\substack{\psi=\psi^{*} \\ g=g^{*}}}+\left.\left.\frac{\partial \hat{\ell}}{\partial m}\right|_{\substack{\psi=\psi^{*} \\ g=g^{*}}} \frac{\partial m^{*}}{\partial g}\right|_{\substack{\psi=\psi^{*} \\ g=g^{*}}}\right]-q^{\prime}\left(g^{*}\right)=0 \tag{B.33}
\end{equation*}
$$

Since $m^{*}=\bar{m}(g, \psi)$, it follows that

$$
\begin{equation*}
\frac{\partial m^{*}}{\partial g}=\frac{1}{r-\mu}-\psi q^{\prime}(g) \tag{B.34}
\end{equation*}
$$

Moreover, since $m^{*}=\bar{m}(g, \psi)$ we know that $\hat{\ell}=\bar{\ell}$ (see Lemma 6). Therefore,

$$
\begin{equation*}
\left.\frac{\partial \hat{\ell}}{\partial g}\right|_{\substack{\psi=\psi^{*} \\ g=g^{*}}}+\left.\left.\frac{\partial \hat{\ell}}{\partial m}\right|_{\substack{\psi=\psi^{*} \\ g=g^{*}}} \frac{\partial m^{*}}{\partial g}\right|_{\substack{\psi=\psi^{*} \\ g=g^{*}}}=\frac{-p(\bar{\ell}) \bar{\ell}+\psi^{*} q^{\prime}\left(g^{*}\right)+\frac{1}{r-\mu}-\psi^{*} q^{\prime}\left(g^{*}\right)}{p(\bar{\ell})\left(1+g^{*}\right)+p^{\prime}(\bar{\ell})\left(\bar{\ell}\left(1+g^{*}\right)-\ell\right)}=0 \tag{B.35}
\end{equation*}
$$

where the last equality follows from the fact that $p(\bar{\ell}) \bar{\ell}=1 /(r-\mu)$ (see Lemma 5 ). Thus, the first-order condition determining $g^{*}$ simplifies to

$$
\begin{equation*}
\frac{1}{r-\mu}-q^{\prime}\left(g^{*}\right)=0 \tag{B.36}
\end{equation*}
$$

implying that $g^{*}=g^{u}$. Thus, we conclude that if $\kappa=\infty$ then the solution to the equity holders' problem is given by $g^{*}=g^{u}, \psi^{*} \in[0,1]$, and $m^{*}=\bar{m}\left(g^{u}, \psi^{*}\right)$.

Next, we investigate for which $\kappa$ the above unconstrained solution is feasible. This is the case if $\kappa \geq \bar{m}\left(g^{u}, \psi\right)$ for all $\psi \in[0,1]$. Note that $\bar{m}\left(g^{u}, \psi\right)$ is decreasing in $\psi$. Therefore, if we define $\bar{\kappa} \equiv \bar{m}\left(g^{u}, 0\right)$ then for all $\kappa \geq \bar{\kappa}$ the "unconstrained solution" is attainable.

Before proceeding further, note for all $\kappa \geq \bar{\kappa}$ equity holders' payoff is given by

$$
\begin{equation*}
v^{*}(\ell)=\frac{1+g^{u}}{r-\mu}-\frac{\eta}{1+\eta} \ell-q\left(g^{u}\right) \tag{B.37}
\end{equation*}
$$

(B.37) shows us that the equity holders capture all the value of new investment and extract as much as possible from the old debt holders given the constraint that $\hat{\ell} \leq \bar{\ell}$. Moreover, note that the post-investment value of equity is independent of $m, \psi$, and $\hat{\ell}$.

Next, consider a situation where $\kappa<\bar{m}\left(g^{u}, 0\right)$ but $\kappa \geq \bar{m}\left(g^{u}, 1\right)$. In this case, the unconstrained solution characterized above is not feasible for some choices of $\psi$. However, the equity holders can still attain the payoff defined in (B.37) by choosing $g^{*}=g^{u}, \psi^{*} \in\left[\underline{\psi}_{\kappa}, 1\right], m^{*}=\bar{m}\left(g^{*}, \psi^{*}\right)$, where $\underline{\psi}_{\kappa}$ is the unique solution to

$$
\begin{equation*}
\kappa=\bar{m}\left(g^{u}, \underline{\psi}_{\kappa}\right) \tag{B.38}
\end{equation*}
$$

Therefore, if we define $\underline{\kappa} \equiv \bar{m}\left(g^{u}, 1\right)$ then the above discussion implies that for all $\kappa \geq \underline{\kappa}$ the equity holders invest the first-best amount. Finally, note that from the definition of $\bar{m}\left(g^{u}, 1\right)$ we have

$$
\begin{equation*}
\frac{\partial \underline{\kappa}}{\partial \ell}<0, \quad \frac{\partial \underline{\kappa}}{\partial \sigma^{2}}>0, \quad \frac{\partial \underline{\kappa}}{\partial r}<0 \tag{B.39}
\end{equation*}
$$

It remains to determine the equity holders' choices when $0<\kappa<\underline{\kappa}$ (the case of $\kappa=0$ is covered by Proposition 4). We first argue, by contradiction, that in this case the equity holders' optimal choices $\left\{g^{*}, \psi^{*}, m^{*}\right\}$ are such that $m^{*}=\kappa<\bar{m}\left(g^{*}, \psi^{*}\right)$. To see this assume, to the contrary, that $\left\{g^{*}, \psi^{*}, m^{*}\right\}$ are such that $m^{*}=\bar{m}\left(g^{*}, \psi^{*}\right)<\kappa$. Then, from Lemma 6 we know that $\hat{\ell}\left(g^{*}, \psi^{*}, m^{*}\right)=$ $\bar{\ell}$, which implies that the equity holders' payoff is given by

$$
\begin{equation*}
\frac{1+g^{*}}{r-\mu}-\frac{\eta}{1+\eta} \ell-q\left(g^{*}\right) \tag{B.40}
\end{equation*}
$$

Note that $\frac{1+g}{r-\mu}-q(g)$ is strictly increasing in $g$ for all $g<g^{u}$ and strictly decreasing in $g$ for all
$g>g^{u}$ and recall that since $\kappa<\underline{\kappa}$ it must be the case that $g^{*} \neq g^{u}$. Furthermore, note that if $g^{*}<g^{u}$ then the equity holders would have incentives to increase their investment from $g^{*}$ to $g^{*}+\varepsilon$ for small $\varepsilon>0$. This is feasible by setting

$$
\begin{equation*}
m^{\prime}=\frac{1+\left(g^{*}+\varepsilon\right)}{r-\mu}-\frac{\eta}{1+\eta} \ell-\psi^{*} q\left(g^{*}+\varepsilon\right) \tag{B.41}
\end{equation*}
$$

as long as $\varepsilon$ is small enough so that $m^{\prime} \leq \kappa$. Hence $g<g^{u}$ cannot be optimal. By a similar argument, if $g^{*}>g^{u}$ then the equity holders would find it optimal and feasible to decrease their investment. Thus, we conclude that a choice of $\left\{g^{*}, \psi^{*}, m^{*}\right\}$ such that $m^{*}=\bar{m}\left(g^{*}, \psi^{*}\right)<\kappa$ is not optimal.

Next, suppose that $\left\{g^{*}, \psi^{*}, m^{*}\right\}$ are such that $m^{*}=\bar{m}\left(g^{*}, \psi^{*}\right)=\kappa$. In this case, the budget constraint implies that

$$
\begin{equation*}
\kappa=\frac{1+g^{*}}{r-\mu}-\frac{\eta}{1+\eta} \ell-\psi^{*} q\left(g^{*}\right) \tag{B.42}
\end{equation*}
$$

while the equity holders' payoff is given by

$$
\begin{equation*}
\frac{1+g^{*}}{r-\mu}-\frac{\eta}{1+\eta} \ell-q\left(g^{*}\right) \tag{B.43}
\end{equation*}
$$

Using (B.42) in (B.43) we see that the equity holders' payoff can be expressed as

$$
\begin{equation*}
\kappa-\left(1-\psi^{*}\right) q\left(g^{*}\right) \tag{B.44}
\end{equation*}
$$

We now argue that the equity holders can attain a strictly higher payoff than the payoff in (B.44) by choosing $\psi^{\prime}=1, m^{\prime}=\kappa$, and an investment $g^{\prime}$ such that $g^{\prime}$ solves the budget constraint

$$
\begin{equation*}
p(\hat{\ell})\left(\hat{\ell}\left(1+g^{\prime}\right)-\ell\right)=\kappa+q\left(g^{\prime}\right) \tag{B.45}
\end{equation*}
$$

with $\hat{\ell}\left(g^{\prime}, \psi^{\prime}, \kappa\right)<\bar{\ell}$. If equity holders make such choices then their payoff would be given by

$$
\begin{equation*}
\frac{1+g^{\prime}}{r-\mu}-p(\hat{\ell}) \ell-q\left(g^{\prime}\right)=\frac{1+g^{\prime}}{r-\mu}-p(\hat{\ell}) \hat{\ell}\left(1+g^{\prime}\right)+\kappa>\kappa \tag{B.46}
\end{equation*}
$$

where the first equality follows from (B.45), while the final inequality follows from the observation that $p(\hat{\ell}) \hat{\ell}<\frac{1+g}{r-\mu}$ for all $\hat{\ell}<\bar{\ell}$ (see Lemmas 1 and 5). Wwe conclude that choice of $\left\{g^{*}, \psi^{*}, m^{*}\right\}$ such that $m^{*}=\bar{m}\left(g^{*}, \psi^{*}\right)=\kappa$ cannot be optimal. It follows that we must have $m^{*}=\kappa<\bar{m}\left(g^{*}, \psi^{*}\right)$.

We argued above that if $\kappa \in(0, \underline{\kappa})$ then $m^{*}=\kappa<\bar{m}\left(g^{*}, \psi^{*}\right)$. Therefore, we have

$$
\begin{equation*}
\left.\frac{\partial m^{*}}{\partial \psi}\right|_{\substack{g=g^{*} \\ \psi=\psi^{*}}}=0 \quad \text { and }\left.\quad \frac{\partial m^{*}}{\partial g}\right|_{\substack{g=g^{*} \\ \psi=\psi^{*}}}=0 \tag{B.47}
\end{equation*}
$$

This implies that, when $\kappa<\underline{\kappa}$, the first-order conditions that determine equity holders' choices of $g$ and $\psi$ are identical to those when $\kappa=0$. Thus, using the same argument as in the proof of Proposition 4 we conclude that $g^{*}>g^{u}$ and $\psi^{*}=1$. This concludes the proof.

## B. 4 Choices of Leverage

We show that increasing leverage requires a larger investment as $\ell$ increases. Since the cost of investment is a strictly convex function and is borne by equity holders (see Proposition 3), this result implies that the cost of increasing leverage increases with $\ell$. We begin by establishing a preliminary result.

Lemma 8. Suppose that investment is financed fully with debt. Then $\partial \hat{\ell} / \partial g \geq(>) 0$ if and only if $g \geq(>) g_{0}$, where $g_{0}$ is the unique solution to $p(\hat{\ell}) \hat{\ell}-q^{\prime}(g)=0$.

Proof. From the expression for $\frac{\partial \hat{\ell}}{\partial g}$ when $\psi=1$ (see (B.19)) and Lemma Lemma 4 we know that $\partial \hat{\ell} / \partial g \geq 0$ if and only if $p(\hat{\ell}) \hat{\ell}-q^{\prime}(g) \leq 0$. At $g=0$ we have $q^{\prime}(g)=0$ and, thus, for small values of $g$ we have $\partial \hat{\ell} / \partial g<0$. Similarly, for sufficiently high $g$ we have $\partial \hat{\ell} / \partial g<0$ since $\lim _{g \rightarrow \infty} q^{\prime}(g)=\infty$. Finally, let $g_{0}$ be a solution to $p(\hat{\ell}) \hat{\ell}-q^{\prime}(g) \leq 0$. Then at $g=g_{0}$ we have

$$
\begin{equation*}
\frac{\partial}{\partial g}\left[p(\hat{\ell}) \hat{\ell}-q^{\prime}(g)\right]=\left[p^{\prime}(\hat{\ell}) \hat{\ell}+p(\hat{\ell})\right] \frac{\partial \hat{\ell}}{\partial g}-q^{\prime \prime}(g)=-q^{\prime \prime}(g)<0 \tag{B.48}
\end{equation*}
$$

implying that $g_{0}$ is unique.
Lemma 9. Let $\bar{g}(\ell)$ be the investment such that at $g=\bar{g}(\ell)$ we have $\hat{\ell}(\bar{g}, \ell)=\ell$. Then, $\bar{g}$ is unique and $\partial \bar{g} / \partial \ell>0$.

Proof. From Lemma 8 we know that there exists a unique $g_{0}$ such that for all $g<g_{0}$ we have $\partial \hat{\ell} / \partial g<0$. Therefore, it has to be the case that $\bar{g}>g_{0}$ and so $\partial \hat{\ell} /\left.\partial g\right|_{g=\bar{g}}>0$.

Now, applying Implicit Function Theorem to $\hat{\ell}(\bar{g}, \ell)=\ell$, we obtain

$$
\begin{equation*}
\frac{\partial \bar{g}}{\partial \ell}=-\frac{\left.\frac{\partial \hat{\ell}}{\partial \ell}\right|_{g=\bar{g}}-1}{\left.\frac{\partial \hat{\ell}}{\partial g}\right|_{g=\bar{g}}} \tag{B.49}
\end{equation*}
$$

Since, $\left.\frac{\partial \hat{\ell}}{\partial g}\right|_{g=\bar{g}}>0$, it is sufficient to determine the sign of the numerator in (B.49). Note that

$$
\begin{equation*}
\frac{\partial \hat{\ell}}{\partial \ell}=\frac{p(\ell)}{p^{\prime}(\hat{\ell})(\hat{\ell}(1+g)-\ell)+p(\hat{\ell})(1+g)}>0 \tag{B.50}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\left.\frac{\partial \hat{\ell}}{\partial \ell}\right|_{g=\bar{g}}-1=\frac{p(\hat{\ell})-p^{\prime}(\hat{\ell})(\hat{\ell}(1+g)-\ell)-p(\hat{\ell})(1+g)}{p^{\prime}(\hat{\ell})(\hat{\ell}(1+g)-\ell)+p(\hat{\ell})(1+g)}=\frac{-p^{\prime}(\hat{\ell}) \hat{\ell} g-p(\hat{\ell}) g}{p^{\prime}(\hat{\ell})(\hat{\ell}(1+g)-\ell)+p(\hat{\ell})(1+g)}<0 \tag{B.51}
\end{equation*}
$$

where the second equality follows by the fact that at $\bar{g}$ we have $\hat{\ell}=\ell$ and the final inequality follows by Lemma 4 and (B.5). This establishes that $\partial \bar{g} / \partial \ell>0$.

## B. 5 Finite-maturity Debt

In this section, we show that our results naturally extend to the case when debt has finite maturity. In particular, we establish that Propositions 4 and 5 continue to hold in this setup. To model finite maturity debt, we follow Leland (1998) and consider debt that has no stated maturity but is continuously retired at par at a constant fractional rate $\xi$. That is, at each instance of time fraction $\xi$ of existing debt matures. It follows that $1 / \xi$ is the average maturity of debt and higher $\xi$ is associated with shorter average maturity. Each unit of debt pays a constant coupon rate of 1 and has the face value $F$. Finally, as in Leland (1998), we assume that equity holders are committed to always rollover their debt (i.e., keep leverage fixed), except possibly at the time of investment. ${ }^{27,28}$

Proposition 7. Propositions 4 and 5 continue to hold in the setup with finite maturity debt.
Proof. Let $T$ be a random default time and assume there are $K$ units of debt outstanding. First, we compute equity holders' liability (i.e., the PDV of equity holders' promises).

$$
\begin{equation*}
L=\left[\int_{0}^{\infty} e^{(r+\xi) t}(1+\xi F) d t\right] K=\frac{1+\xi F}{r+\xi} K=\varrho \frac{K}{r} \tag{B.52}
\end{equation*}
$$

where

$$
\begin{equation*}
\varrho \equiv \frac{r(1+\xi F)}{r+\xi} \tag{B.53}
\end{equation*}
$$

and $\varrho \rightarrow 1$ as $\xi \rightarrow 0$. Therefore, $K=\frac{r L}{\varrho}$. In what follows, we use $L$ as the state variable to make analysis easily comparable to the analysis in the main paper.

The price of finite-maturity debt is given by

$$
\begin{equation*}
P(Z, L ; \xi)=\mathbb{E}\left[\int_{0}^{T} e^{-(r+\xi) t}(1+\xi F) d t+e^{-(r+\xi) T} \frac{\varrho V^{D}}{r L}\right] \tag{B.54}
\end{equation*}
$$

where $V^{D}$ is the value of the firm at default. Suppose that $\underline{Z}$ is the value of $Z$ at which firm default so that $V^{D}=\frac{\underline{Z}}{r-\mu}$. Then,

$$
\begin{equation*}
P(Z, L ; \xi)=\frac{1+\xi F}{r+\xi}\left[1-\left(\frac{\underline{Z}}{Z_{0}}\right)^{\eta}\right]+\frac{\varrho \underline{Z}}{(r-\mu) r L}\left(\frac{\underline{Z}}{Z_{0}}\right)^{\eta} \tag{B.55}
\end{equation*}
$$

where

$$
\begin{equation*}
\eta \equiv \frac{\mu-\frac{1}{2} \sigma^{2} \sqrt{\left(\mu-\frac{1}{2} \sigma^{2}\right)^{2}+2(r+\xi) \sigma^{2}}}{\sigma^{2}} \tag{B.56}
\end{equation*}
$$

[^19]As $\xi \rightarrow 0$ the above price converges to the price in the baseline model.
Next, we consider the total value of the firm (total enterprise value). We have

$$
\begin{equation*}
T E V=\mathbb{E}\left[\int_{0}^{\infty} e^{-r t} Z_{t} d t\right]=\frac{Z}{r-\mu} \tag{B.57}
\end{equation*}
$$

Since equity holders are residual owners, it follows that the value of equity is given by

$$
\begin{equation*}
V(Z, L)=T E V-P(Z, L) \frac{r L}{\varrho}=\frac{Z_{0}}{r-\mu}-\left[1-\left(\frac{\underline{Z}}{Z_{0}}\right)^{\eta}\right] L-\frac{\underline{Z}}{r-\mu}\left(\frac{\underline{Z}}{Z_{0}}\right)^{\eta} \tag{B.58}
\end{equation*}
$$

It remains to determine the optimal default threshold. We know that $V(Z, L)$ has to satisfy the smooth-pasting condition and, hence,

$$
\begin{equation*}
0=V_{Z}(\underline{Z}, L)=\frac{1}{r-\mu}-\left[L-\frac{\underline{Z}}{r-\mu}\right] \eta \frac{1}{\underline{Z}} \tag{B.59}
\end{equation*}
$$

Solving the above equation for $\underline{Z}$, we obtain

$$
\begin{equation*}
\underline{Z}=\frac{\eta}{1+\eta}(r-\mu) L \tag{B.60}
\end{equation*}
$$

Using the expression for $\underline{Z}$ found above in the expression for the value of equity (B.58) and simplifying, we obtain

$$
\begin{equation*}
V(Z, L)=\frac{Z}{r-\mu}-\left(1-\chi \frac{1}{1+\eta}\left(\frac{L}{Z}\right)^{\eta}\right) L \tag{B.61}
\end{equation*}
$$

where $\chi$ is defined in (8). Thus, as in the baseline model, we have $V(Z, L)=v(\ell) Z$, where

$$
\begin{equation*}
v(\ell)=\frac{1}{r-\mu}+\left(1-\chi \frac{1}{1+\eta} \ell^{\eta}\right) \ell \tag{B.62}
\end{equation*}
$$

Note that the expression for $v(\ell)$ in (B.62) is the same as in (11). Therefore, conditional on the PDV of equity holders' liabilities, the equity holders' default decision is unchanged.

Similarly, using the expression for $\underline{Z}$ we can simplify the expression for $P(Z, L ; \xi)$. In particular, we have

$$
\begin{equation*}
P(Z, L ; \xi)=\left(1-\chi \frac{1}{1+\eta} \ell^{\eta}\right) \frac{\varrho}{r} \tag{B.63}
\end{equation*}
$$

It remains to consider the budget constraint that equity holders face at the time of investment. As in the baseline model, let $\hat{L}$ denote post-investment liabilities. As we discussed above, if equity holders issue $K^{\prime}$ units of debt to finance their investment $g$ then the associated increase in liabilities is given by $\frac{r}{\varrho}(\hat{L}-L)$, where $\hat{L}=\frac{\varrho}{r}\left(K^{\prime}+K\right)$, and the funds the equity holders obtain are equal to

$$
\begin{equation*}
\left(1-\chi \frac{1}{1+\eta} \ell^{\eta}\right)(\hat{L}-L) \tag{B.64}
\end{equation*}
$$

Therefore, the equity holders' budget constraint (divided by $Z$ ) is given by

$$
\begin{equation*}
\left(1-\chi \frac{1}{1+\eta} \ell^{\eta}\right)(\hat{\ell}(1+g)-\ell)=\psi q(g)+m \tag{B.65}
\end{equation*}
$$

We see that compared to the baseline model, the only difference is that $\eta$ now depends on $\xi$. Thus, Propositions 4 and 5 remain valid in the model with finite maturity debt.

## B. 6 Cash holdings

Suppose the firm is allowed to hold cash on its balance sheet. One may wonder whether the firm can use an increase in its cash holding as a way to dilute existing debt holders instead of resorting to inefficient investment. In this section, we show that this is not the case and that equity holders would not have incentives to raise cash in our baseline model.

To see this, consider our baseline model with no bankruptcy costs and $\kappa=0$ (i.e, no equity payouts) but assume that equity holders at time 0 can raise cash. ${ }^{29}$ Furthermore assume that cash (1) has to be left on firm's balance sheet (i.e., equity holders cannot withdraw it/use it) and (2) is distributed among all debt holders on pari passu basis at the time of default. We also assume that cash earns zero rate of return. ${ }^{30}$ Note that under these assumptions equity holders do not benefit directly from raising cash, only indirectly via increase in leverage. It follows that conditional on leverage, the value of equity and equity holders' optimal default decisions are unchanged. Initially the firm is assumed to hold no cash.

Let $P(L, Z, C)$ denote the price of unit of debt issued by firm with liability $L$, cash flows, $Z$, and cash holdings, $C$. Then,

$$
\begin{equation*}
P(L, Z, C)=\frac{1}{r}\left[\left(1-\chi \frac{1}{1+\eta} \ell^{\eta}\right)+\frac{C}{L} \chi \ell^{\eta}\right]=\frac{p(\ell, c)}{r}, \tag{B.66}
\end{equation*}
$$

where $\frac{C}{L} \chi \ell^{\eta}$ captures the present value of cash that a debt holder of a unit of debt will receive at the time of default and $c=C / Z$. It follows that the value of equity of a firm with cash flows $Z$, liability $L$, and cash holding $C$ can be written as

$$
\begin{equation*}
V(\ell, Z)=\left[\frac{1}{r-\mu}-p(\ell, 0) \ell\right] Z \tag{B.67}
\end{equation*}
$$

That $V(\ell, Z)$ is independent of $C$ follows from the observation that only care about the value of cash flows that debt holders were promised to obtain.

Now suppose that at time 0 equity holders decide how much cash to raise. Let $\hat{c}$ and $\hat{\ell}$ be firm's cash holdings and leverage after the firm raised cash. Then, just after the equity holders made their

[^20]decisions, the value of equity is given by
\[

$$
\begin{equation*}
V(\hat{\ell}, Z)=\left[\frac{1}{r-\mu}-p(\hat{\ell}, 0) \hat{\ell}\right] Z \tag{B.68}
\end{equation*}
$$

\]

The equity holder's budget constraint is given by

$$
\begin{equation*}
p(\hat{\ell}, \hat{c})(\hat{\ell}-\ell)=\hat{c} \tag{B.69}
\end{equation*}
$$

Note that (B.69) implies that $\partial \hat{\ell} / \partial \hat{c}>0$. Now, differentiating the value of equity in (B.68) w.r.t. $\hat{c}$ we obtain

$$
\begin{equation*}
\frac{\partial V(\hat{\ell}, Z)}{\partial \hat{c}}=\frac{\partial \hat{\ell}}{\partial \hat{c}}\left(\frac{p(\hat{\ell}, 0)}{\partial \hat{\ell}} \hat{\ell}+p(\hat{\ell}, 0)\right)=-\frac{\partial \hat{\ell}}{\partial \hat{c}}\left[1-\chi \hat{\ell}^{\eta}\right]<0 \tag{B.70}
\end{equation*}
$$

where the last inequality follows the fact that $1-\chi \hat{\ell}^{\eta}>0$ (see Corollary 3 ). This establishes that the value of equity is strictly decreasing in $\hat{c}$ and so equity holders have no incentives to raise cash. We summarize the above result in the following lemma.

Lemma 10. The equity holders never choose to raise cash.
Why do equity holders have incentive to engage in inefficient investment but do not have incentive to raise cash? To understand this note that when equity holders raise cash they do decrease the value of existing coupon claims. In particular, in contrast to inefficient investment, equity holders only benefit indirectly from raising cash.

Moreover, raising cash also benefits existing debt holders by moving forward the timing of default and even increasing the value of their default claims by providing additional cash in default. ${ }^{31}$ This limits the dilution mechanism. It also implies that new debt holders will recover part of the funds they provided to equity holders. Therefore, new debt holders require high compensation for their funds. As such the cost of raising cash exceeds the benefit due to dilution of existing debt holders.

Instead, when equity holders engage in inefficient investment the equity holders also benefit from it directly via higher cash flows the firm earns till the time of default. This direct benefit together with indirect benefit due to dilution of existing debt holders makes equity holders overinvest. If investment was costly but unproductive (i.e., had no effect on cash flows) then equity holders would never make such investment for similar reasons why they do not raise cash (i.e., indirect benefit via dilution being too small to compensate for the cost of raising funds via debt).

## Appendix C Repeated Investment Derivations

This section derives the ODEs for a repeated investment decisions, which introduces a controlled jump-process. Assume that upon an arrival of an financing-investment opportunity, the state jumps

[^21]to a deterministic function of the current state, $\hat{\ell}(\ell)$. Define the jump size as $\tilde{g}(\ell) \equiv \hat{\ell}(\ell)-\ell$. Then the SDE for $\ell$ is
\[

$$
\begin{equation*}
d \ell_{t}=\left(\sigma^{2}-\mu\right) \ell_{t} \mathrm{~d} t+\sigma \ell_{t} d \mathbb{W}_{t}+(\hat{\ell}(\ell)-\ell) d \mathbb{N}_{t} \tag{C.1}
\end{equation*}
$$

\]

where $\mathbb{N}_{t}$ is a homogeneous Poisson process with arrival rate $\lambda \geq 0$.

Firm's HJBE First, we will derive the HJBE in $\ell$-space without the jumps, and add them. Set $V(Z, L)=Z v(L / Z)$ and differentiate w.r.t. $Z$

$$
\begin{align*}
\boldsymbol{\partial}_{Z} V(Z, L) & =v(L / Z)-\frac{L}{Z} \boldsymbol{\partial}_{\ell} v(L / Z)=v(\ell)-\ell \boldsymbol{\partial}_{\ell} v(\ell)  \tag{C.2}\\
\boldsymbol{\partial}_{Z Z} V(Z, L) & =\frac{L^{2}}{Z^{3}} \boldsymbol{\partial}_{\ell \ell} v(L / Z)=\frac{1}{Z} \ell^{2} \boldsymbol{\partial}_{\ell \ell} v(\ell) \tag{C.3}
\end{align*}
$$

Use the ODE in (5), divide by $Z$, and use the above derivatives to obtain

$$
\begin{equation*}
(r-\mu) v(\ell)=1-r \ell-\mu \ell \boldsymbol{\partial}_{\ell} v(\ell)+\frac{\sigma^{2}}{2} \ell^{2} \boldsymbol{\partial}_{\ell \ell} v(\ell) \tag{C.4}
\end{equation*}
$$

Default Decision The notation denotes $\left.\right|_{\ell}$ as the evaluation of a function at $\ell$.
For the firm's boundary conditions, add in artificial reflecting barriers at some $\ell_{\min }$ and $\ell_{\max }$. We will ensure that the equilibrium $\ell_{\min }<\bar{\ell}$ so it is never binding in the solution, the $\ell_{\max }$ will be chosen large enough to not effect the solution.

Then, for the firm, we can write the DVI for their stopping problem as

$$
\begin{align*}
u(c) & \equiv 1-r \ell  \tag{C.5}\\
\mathcal{L}_{v} & \equiv r-\mu+\mu \ell \boldsymbol{\partial}_{\ell}-\frac{\sigma^{2}}{2} \ell^{2} \boldsymbol{\partial}_{\ell \ell}-\lambda\left(\left.\cdot\right|_{\ell+\tilde{g}(\ell)}-\left.\cdot\right|_{\ell}\right)  \tag{C.6}\\
0 & =\min \left\{\mathcal{L}_{\ell} v(\ell)-u(\ell), v(\ell)\right\}  \tag{C.7}\\
\boldsymbol{\partial}_{\ell} v\left(l_{\min }\right) & =0  \tag{C.8}\\
\boldsymbol{\partial}_{\ell} v\left(l_{\max }\right) & =0 \tag{C.9}
\end{align*}
$$

We would numerically find a $\bar{\ell}$ which fulfills the indifference point, and then find the value of liquidation per unit of PV of liabilities is

$$
\begin{equation*}
v^{\mathrm{liq}} \equiv(1-\theta) \frac{\lim _{\ell \rightarrow 0} v(\ell)}{\bar{\ell}} \tag{C.10}
\end{equation*}
$$

Bond Pricing The price of a bond, $P(Z, L)$ pays 1 unit until default. The ODE in the continuation region without jumps is

$$
\begin{equation*}
r P(Z, L)=1+\mu Z \boldsymbol{\partial}_{Z} P(Z, L)+\frac{\sigma^{2}}{2} Z^{2} \boldsymbol{\partial}_{Z Z} P(Z, L) \tag{C.11}
\end{equation*}
$$

Take the defintion $r P(Z, L) \equiv p(L / Z)$ and differentiate with respect to $Z$

$$
\begin{align*}
r \boldsymbol{\partial}_{Z} P(Z, L) & =-\frac{1}{Z} \ell \boldsymbol{\partial}_{\ell} p(\ell)  \tag{C.12}\\
r \boldsymbol{\partial}_{Z Z} P(Z, L) & =\frac{1}{Z^{2}}\left(2 \ell \boldsymbol{\partial}_{\ell} p(\ell)+\ell^{2} \boldsymbol{\partial}_{\ell \ell} p(\ell)\right) \tag{C.13}
\end{align*}
$$

Multiply by $r$ and substitute the derivatives into (C.11)

$$
\begin{equation*}
r p(\ell)=r+\left(\sigma^{2}-\mu\right) \ell \boldsymbol{\partial}_{\ell} p(\ell)+\frac{\sigma^{2}}{2} \ell^{2} \boldsymbol{\partial}_{\ell \ell} p(\ell) \tag{C.14}
\end{equation*}
$$

In default, the bond is entitled to a share of $r L$ units of the liquidation value $V((1-\theta) \underline{Z}(L), 0)$, hence $P(\bar{\ell})=\frac{V((1-\theta) \underline{Z}(L), 0)}{r L}$. Divide by $r$ and use the definitions of $v^{\text {liq }}$ and $p(\cdot)=P(\cdot) / r$ to find that the boundary condition is $p(\bar{\ell})=v^{\text {liq }}$.

Summarizing, bond pricers take $v^{\mathrm{liq}}$ and $\bar{\ell}$ as given, and then solve

$$
\begin{align*}
\mathcal{L}_{p} & \equiv r-\left(\sigma^{2}-\mu\right) \ell \boldsymbol{\partial}_{\ell}-\frac{\sigma^{2}}{2} \ell^{2} \boldsymbol{\partial}_{\ell \ell}-\lambda\left(\left.\cdot\right|_{\ell+\tilde{g}(\ell)}-\left.\cdot\right|_{\ell}\right)  \tag{C.15}\\
\mathcal{L}_{p} p(\ell) & =r  \tag{C.16}\\
\boldsymbol{\partial}_{\ell} p\left(\ell_{\min }\right) & =0  \tag{C.17}\\
p(\bar{\ell}) & =v^{\text {liq }} \tag{C.18}
\end{align*}
$$

where the lower boundary is an artificial reflecting barrier and the upper boundary is the liquidation absorbing barrier.

Investment Finally, the objective function of the firm at every arrival point $\lambda$ remains to maximize the equity value. Given an equilibrium $p(\ell)$ and $v(\ell)$ functions-consistent with the optimal jump process, the agent solves the problem described by (20)-(22).

First-Best The first-best is derived through a guess-and-verify approach. First, guess that the user would choose a constant $g$ due to the homotheticity of the problem. With that, the unnormalized Bellman equation (with jumps) is

$$
\begin{equation*}
r V(Z)=Z+\mu Z V^{\prime}(Z)+\frac{\sigma^{2}}{2} Z^{2} V^{\prime \prime}(Z)+\lambda \max _{g}\left\{V((1+g) Z)-V(Z)-\zeta \frac{g^{2}}{2}\right\} \tag{C.19}
\end{equation*}
$$

Take the first-order condition

$$
\begin{equation*}
\zeta g Z=Z V^{\prime}((1+g) Z) \tag{C.20}
\end{equation*}
$$

Guess the solution to the problem is $V(Z)=A Z$ for an undetermined $Z$, and subtitute into the (C.19) and solve for $A$ to find,

$$
\begin{equation*}
A=\frac{1-\frac{1}{2} \zeta g^{2} \lambda}{-g \lambda-\mu+r} \tag{C.21}
\end{equation*}
$$

Similarly, substitute the guess into (C.20) to find $g=\frac{A}{\zeta}$. Use this expression to eliminate $A$ in (C.21), solve the quadratic for $g$, and choose the positive root to find,

$$
\begin{equation*}
g^{u}=\frac{1}{\zeta(r-\mu)\left(\frac{1}{2}\left(\sqrt{1-\frac{2 \lambda}{\zeta(r-\mu)^{2}}}-1\right)+1\right)}=\frac{2}{\sqrt{\zeta\left(\zeta(r-\mu)^{2}-2 \lambda\right)}+\zeta(r-\mu)} \tag{C.22}
\end{equation*}
$$

In addition, given that $V(Z)=A Z$ and noting that $V(Z, 0)$ we obtain $v(0)=\frac{V(Z, 0)}{Z}=A$. Consequently, for a default threshold $\bar{\ell}$, the liquidation value per unit of defaultable console in (C.10) is,

$$
\begin{equation*}
v^{\mathrm{liq}}=\frac{1-\theta}{\bar{\ell}} \frac{1-\frac{1}{2} \zeta\left(g^{u}\right)^{2} \lambda}{r-\mu-g^{u} \lambda} \tag{C.23}
\end{equation*}
$$

When $\lambda=0$, these all nest the $g^{u}=\frac{1}{\zeta(r-\mu)}$ case.

## Appendix D Covenants

There is a large existing literature that documents that covenants are commonly used to protect existing debtholders (see Smith Jr and Warner (1979), Billett et al. (2007), Chava et al. (2010), Reisel (2014)). ${ }^{32}$ Below, we discuss how commonly used covenants would affect our results. Our main conclusion is that, depending on their type, covenants are unlikely to resolve the issues and even further support our modeling choices.

Restrictions on payouts: We think of restrictions on payouts being captured by our model parameter $\kappa$. From Figure 5 we see that as these restrictions become tighter, they tend to mitigate underinvestment for low leverage firms but tend to exacerbate overinvestment for the remaining firms. While it is theoretically possible to devise payout restrictions that would restore first-best investment, such covenants would have to be state-contingent. It is therefore unlikely that optimality could be restored through such covenants in practice.

Secured debt restrictions: Secured debt restrictions (often referred to as negative pledge covenants) prohibit firms from issuing secured debt, unless all pre-existing debt also obtains a proportional claim to the same collateral. This covenant is typically used by unsecured lenders to protect themselves from dilution, though it may be difficult to enforce in practice (see Donaldson et al. (2019)). In our benchmark model, all debt already has equal priority claims in bankruptcy, thus this type of covenant does not have any bite.

Restrictions on leverage: Restrictions on leverage are relatively common covenants particularly for non-investment grade firms (see Billett et al. (2007)). However, in our model investment distortions occur even at low to medium leverage levels (firms that are unlikely to violate leverage restrictions). Moreover, the highest leverage firms in our model tend to even decrease leverage while overinvesting. It is therefore unlikely that covenants that restrict leverage can correct equity holders' investment

[^22]incentives in our model.
Senior debt restrictions: Senior debt restrictions prohibit the firms from issuing senior debt. These types of covenants are empirically extremely rare affecting only $0.2 \%$ of firms in Billett et al. (2007) post-2000 sample. In our model, this would imply that all new debt would have to be junior compared to the existing debt. As we show in the Appendix, this covenant would resolve the issue of overinvestment but at the cost of firms underinvesting for all levels of leverage.

## Appendix E Calibration

To discipline parameters for the exposition, we calibrate to moments from the firm dynamics (Sterk et al. (2021)) and investment spikes (Gourio and Kashyap (2007)) literature.

| Variable | Value | Description |
| :---: | :---: | :---: |
| $r$ | 0.0765 | Discount rate of cash flows on long-term real risk-free rate, measured as the 10-year nominal Treasury rate (from FRED) minus 1-year Survey of Professional Forecasters inflation expectations for the GDP deflator. In addition, since we do not have exit in our model (but empirical investment rates would reflect the exit probability) we add the estimated $4.1 \%$ exogenous exit rate estimated in in Sterk et al. (2021). |
| $\lambda$ | 0.3 | Use investment spikes literature estimates in Gourio and Kashyap (2007) (Table 1 of the NBER version). They find that in close to $30 \%$ of US plant-years have an investment spike of $12 \%$ or more of total assets. In particular, as a proportion of investment relative to assets, $11.6 \%$ invest between 0.12 and $0.2,8 \%$ invest between 0.2 and 0.35 , and $8.3 \%$ invest more than 0.35 ) |
| $(\mu, \sigma, \zeta)$ | $(-0.0514,0.1534,50.036)$ | Jointly solve with Use $g^{u}$ from (34) with $\lambda$, the closed form first-best dynamics $\mathbb{E}\left[d \log Z_{t}\right]=\frac{\partial}{\partial t} E\left[\log \left(\frac{Z_{t}}{Z_{0}}\right)\right]=\mu-\frac{1}{2} \sigma^{2}+\lambda \log \left(1+g^{u}\right)$ and $\mathbb{V}\left[d \log Z_{t}\right] \equiv \mathbb{E}\left[\left(d \log Z_{t}\right)^{2}\right]-\mathbb{E}\left[d \log Z_{t}\right]^{2}=\sigma^{2}+\lambda \log \left(1+g^{u}\right)^{2}$ using targets in Sterk et al. (2021) |

## Table 1: Calibration

To guide the relevant range of $\ell$, Palomino et al. (2019) report an average interest coverage ratio of around 4 for the period 1970-2017. They also find that an interest coverage ratio of 1.5 is associated with an annual default probability of $3 \%$ (i.e. a 5 -year default probability of roughly $15 \%$ ), that $30 \%$ of creditors had an interest coverage ratio of 2 or less, and about $10 \%$ of borrowers had an interest coverage ratio of 1 or less. We therefore consider $\ell(0) \in\{3,9\}$ corresponding to interest coverage ratios of 4 and 1.5 to capture a highly levered and an average firm.


[^0]:    *University of British Columbia, Vancouver School of Economics; and ${ }^{\dagger}$ University of Chicago, Harris School of Public Policy, NBER, and CEPR.
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[^1]:    ${ }^{1}$ While it manifests in various ways, limited liability is a protection of firm owners' (or equity holders) non-firm assets (including human capital) from creditors following a firm's bankruptcy. In all of its forms, the central friction is that limited liability leads to a commitment problem which limits possible contracts. Firm owners raising debt would otherwise promise to either (1) never default, or (2) pay debt holders personal assets outside of the firm-thereby making punishments more effective. Similar frictions, albeit in different institutional context, also exist in the sovereign debt literature, where a lack of commitment to not issue future debt causes sovereigns to inefficiently rely on short-term debt (see Hatchondo et al. (2016) or Aguiar et al. (2019)).

[^2]:    ${ }^{2}$ In contrast to risk-shifting and entrepreneurship models (e.g., Jensen and Meckling (1976) and Vereshchagina and Hopenhayn (2009)), debt in our model can be thought of as being collateralized which distinguishes the mechanism from standard channels of dilution. Moreover, the incentive to overinvest in our model applies to the wide range of firms that are still well away from their default threshold, and not just currently financially distressed. Since dilution is driven by a change in default timing, it is different from standard debt dilution in two-period models (Fama and Miller (1972)). Thus, as we argue, it cannot be solved with covenants securing existing debt holders' claims in bankruptcy.

[^3]:    ${ }^{3}$ In the model, we assume that firm owners face a constraint that limits the issuance of debt for equity payouts. This constraint is motivated by common debt covenants that impose dividend and equity payout restrictions (Billett et al. (2007)).

[^4]:    ${ }^{4}$ DeMarzo and He (2021) also consider an extension of their model with real investment. However, because in their model with investment, there is no recovery value of the firm in default, they obtain a classic debt overhang result where all firms with existing debt underinvest (as in Myers (1977)). In contrast, we focus on the case with a positive recovery value and analyze how indirect dilution provides incentives for overinvestment. The assumption of no recovery value is maintained throughout most of their analysis. As our analysis implies, this is not an innocuous assumption and introduces a time-inconsistency. In addition, empirical studies tend to estimate real costs of default to be relatively small (see, for example, Bris et al. (2006)).

[^5]:    ${ }^{5}$ That is, all claims to the assets of the firm in default are pari passu. While this might seem to make the traditional source of dilution possible (i.e., selling new pari passu debt to dilute the claims to the firm in default) by construction that will not be the case in our model. See Section 2.4 for more details.
    ${ }^{6}$ Due to the absence of fixed costs in this model, cash flows are equivalent to EBITDA profits and proportional to both the assets-in-place and enterprise value. The book value of liabilities $L$ does not take into account the equity holder's option to default. For example, if the firm's liabilities consist of one unit of defaultable consol that promises coupon $c$ every instant of time then $L=\int_{0}^{\infty} c e^{-r s} d s$.

[^6]:    ${ }^{7}$ We interpret $\kappa$ as a restriction of equity holders' choices arising from covenants protecting existing debt holders. Such covenants are common (see Billett et al. (2007)). Alternatively, this restriction can be interpreted as arising from financial regulations that protect existing debt holders. The restriction that $M \geq 0$ is without loss of generality since equity holders would never choose $M<0$ (which, in the model, corresponds to buying back debt).
    ${ }^{8}$ In Appendix B. 5 we show that our results continue to hold when equity holders' finance their investment with debt of finite maturity (modeled as in Chatterjee and Eyigungor (2012) or Leland (1998)).

[^7]:    ${ }^{9}$ We differentiate between pre-investment and post-investment states when discussing the investment decision, in which case we denote the post-investment states by $\hat{L}$ and $\hat{Z}=Z(1+g)$.

[^8]:    ${ }^{10}$ This constraint is similar in spirit to a constraint that limits the ability of the firm to sell assets in distress but before bankruptcy (as is the case in common bankruptcy laws). It captures the observation that existing debt holders would be able to (at least partially) block issuance of large amount of debt just before default.

[^9]:    ${ }^{11}$ The reformulated objective function can be obtained by substituting the budget constraint into the expression for the post-investment value of equity (the RHS of $(20)$ ).

[^10]:    ${ }^{12}$ We discuss how the presence of bankruptcy costs affects investment in Section 3.6.
    ${ }^{13}$ Note that once $\{g, \psi, m\}$ are chosen, $\hat{\ell}$ is determined by the budget condition (21). Thus, we can treat $\hat{\ell}$ as a function of $\{g, \psi, m\}$.

[^11]:    ${ }^{14}$ As we show in Appendix B. 5 this result (as well as other results reported in this section) continue to hold when equity holders finance their debt of finite maturity as in Leland (1998).
    ${ }^{15}$ Jungherr and Schott (2021) show how high leverage may lead to slow recoveries from recessions.

[^12]:    ${ }^{16}$ To be precise, before the investment takes place, the value of existing debt holders' bankruptcy claims at the time of default is $\frac{\eta}{1+\eta} L$, the value of the firm in default (see (13)). After the investment takes place the value of existing debt holders' bankruptcy claims at the time of default is given by $V((1-\theta) \underline{Z}(\hat{L}), 0) \times(L / \hat{L})$, which is also equal to $\frac{\eta}{1+\eta} L$ (since the post-investment firm's value in default $\left(\frac{\eta}{1+\eta} \hat{L}\right)$ is divided proportionally between new and old debt holders).
    ${ }^{17}$ To be more precise, define $\bar{g}$ as the level of investment such that $\hat{\ell} \geq \ell$ if and only if $g \geq \bar{g}$. Then $\bar{g}$ is an increasing function of $\ell$ (see Lemma 9). For high $\ell$, choosing $g>\bar{g}$ (i.e., increasing leverage) is associated with investment cost, $q(g)$, which exceeds potential benefits from dilution.
    ${ }^{18}$ Note that deleveraging occurs not because firms buy back debt, but rather because following investment cash flows increase more than debt (as issuing more debt becomes too costly for highleverage firms). Thus, just as in the recent literature on leverage ratcheting (Admati et al. (2018)), in our model equity holders never have an incentive to buy back debt.

[^13]:    ${ }^{19}$ We view our mechanism as complementary to the underinvestment implied by collateral constraints models. The additional sources of market incompleteness emphasized in that literature most likely are of first-order importance for small and young firms or firms with standardized capital. In contrast, the inefficiency emphasized in our paper, while in principle affecting all firms that are protected by limited liability, is probably most relevant for well-established firms with outstanding debt.

[^14]:    ${ }^{20}$ This suggests that in our setup, equity holders have an incentive to "collude" with new creditors in order to dilute the existing debt holders. A similar mechanism has been emphasized recently by Aguiar et al. (2019) within the context of sovereign default.

[^15]:    ${ }^{21}$ See Appendix E for detailed discussion of other parameters' values.

[^16]:    ${ }^{22}$ To see this note that the last part of Proposition 6 implies that the first-best unconstrained investment is an increasing function of $\lambda$. Since the investment cost is strictly convex and is borne by equity holders, overinvestment is more costly when $\lambda$ is high.

[^17]:    ${ }^{23}$ See the discussion following Proposition 4 and Lemma 9.

[^18]:    ${ }^{24}$ This interpretation is possible since the value of the firm in bankruptcy is known with certainty since investment is deterministic and the post-investment paths for $Z(t)$ are continuous.
    ${ }^{25}$ As in the benchmark model, equity holders' liability are equal to the PDV of promised coupon payments. For that reason, equity holders' liabilities are unchanged if debt is collateralized or carry a proportional claim to the value of the firm in default.
    ${ }^{26}$ The maximum value that equity holders can possibly promise as collateral is the post-investment

[^19]:    ${ }^{27}$ Modeling finite maturity debt in this way leads to analytically tractable problem and has been popular both in corporate finance (see, for example, Leland (1998), DeMarzo and He (2021), He and Xiong (2012)) and in sovereign debt literature (see, for example, Chatterjee and Eyigungor (2012)).
    ${ }^{28}$ The commitment to rolling over debt is a common assumption in this literature. See the discussion in Dangl and Zechner (2021).

[^20]:    ${ }^{29} \mathrm{We}$ are isolating the choice of cash from the other investments because they are easily separable, but it could be a joint decision. We choose the simple $\kappa=0$ case because equity holders always want to maximize direct equity payouts and the presence of cash does not change this incentive.
    ${ }^{30}$ Allowing cash earn low but positive rate of return does not change the result. In addition, using the collateralized debt interpretation rather than parri passu would be identical since the existing collateral claims are not diluted in bankruptcy with our mechanism.

[^21]:    ${ }^{31}$ If default claims were entirely collateralized and old default claims could not benefit from the cash in the firm - as in our collateralized interpretation of the model, the results would be the same. The key is that the equity holders cannot gain from rearranging cashflows between new and old claimants.

[^22]:    ${ }^{32}$ For theoretical analysis of covenants see Smith Jr and Warner (1979), Donaldson et al. (2019, 2020), and references therein.

